

Almost all Steiner triple systems are almost resolvable

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Abstract

We show that for any n divisible by 3, almost all order- n Steiner triple systems admit a decomposition of almost all their triples into disjoint perfect matchings (that is, almost all Steiner triple systems are almost *resolvable*).

1 Introduction

One of the oldest problems in combinatorics, posed by Kirkman in 1850 [24] is the following:

Fifteen young ladies in a school walk out three abreast seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast.

This problem was solved by Kirkman a few years earlier [23] (although the first published solution is due to Cayley [6]).

In order to frame the above question in more generality we need to introduce some terminology. An *order n Steiner Triple System* (**STS**(n) for short) is a collection of triples of $[n] := \{1, \dots, n\}$ for which every pair of elements is contained in exactly one triple. These objects are named after Jakob Steiner, who observed the existence of such systems in 1853 (it is an interesting historical note that this happened after Kirkman proposed his problem).

As an example, observe that the collection $\{123, 145, 167, 246, 257, 347, 356\}$ forms an **STS**(7), and a simple counting argument yields that an **STS**(n) can only exist if $n \equiv (1 \text{ or } 3) \pmod{6}$. It is also known that for every such n , a construction of an **STS**(n) can be achieved. The study of Steiner triple systems (and some natural generalisations, such as Steiner systems and block designs) has a long and rich history. These structures have strong connections to a wide range of different subjects, ranging from group theory, to finite geometry, to experimental design, to the theory of error-correcting codes, and more. For an introduction to the subject the reader is referred to [7].

Now, for an order- n Steiner triple system S , a *matching* in S is a collection of disjoint triples of S , and a *perfect matching* (also known as a *resolution class* or *parallel class*) is a matching covering the entire ground set $[n]$. Observe that a perfect matching consists of exactly $\frac{n}{3}$ triples and therefore can only exist when $n \equiv 3 \pmod{6}$. We say that S is *resolvable* (or has *parallelism*) if the triples in S can be perfectly partitioned into perfect matchings. Since S has $\binom{n}{2}/3$ triples and every perfect matching has $n/3$ edges, such a partition must consist of exactly $\frac{n-1}{2}$ perfect matchings.

Using the above terminology, Kirkman’s problem is simply asking whether there exists a resolvable **STS**(15). More generally, one can ask whether there exists a resolvable **STS**(n) for any $n \equiv 3 \pmod{6}$. This problem was famously solved in the affirmative by Ray-Chaudhuri and Wilson [39] in 1971, over 100 years after Kirkman posed his problem.

Despite the difficulty in proving even that resolvable Steiner triple systems exist, the existence of many large matchings seems to actually be a “typical” property of Steiner triple systems. Early results were based on Rödl-nibble¹ type arguments. A result of Pippenger and Spencer [37] shows that every **STS**(n)

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¹The “nibble” is a type of probabilistic argument introduced by Rödl [40] for finding almost-perfect matchings in set systems. Far beyond the study of Steiner triple systems, it has had significant influence throughout combinatorics in the last 30 years.

contains disjoint matchings of size $n - o(n)$ which cover a $(1 - o(1))$ -fraction of its triples, and Alon, Kim and Spencer [1] later proved that every $\mathbf{STS}(n)$ contains at least one matching which covers all but at most $O(n^{1/2} \log^{3/2} n)$ elements. More recently, building on the breakthrough work of Keevash [19] concerning existence and completion of block designs, Kwan [31] proved that if $n \equiv 3 \pmod{6}$, then almost all (meaning a $(1 - o(1))$ -fraction of) order- n Steiner triple systems have a perfect matching. In fact, almost every $\mathbf{STS}(n)$ has $((1 - o(1))n / (2e^2))^{n/3}$ different perfect matchings. This proof was adapted by Morris [36] to show that almost every $\mathbf{STS}(n)$ has $\Omega(n)$ disjoint perfect matchings. We would go so far as to make the following conjecture:

Conjecture 1.1. *For $n \equiv 3 \pmod{6}$, almost every $\mathbf{STS}(n)$ is resolvable.*

We remark that in Conjecture 1.1, “almost every” certainly cannot be replaced with “every”: Bryant and Horsley [4] proved that for infinitely many $n \equiv 3 \pmod{6}$ there exist Steiner triple systems with not even a single perfect matching.

Of course, Conjecture 1.1 and the aforementioned result of Kwan are really about *random* Steiner triple systems: we are trying to understand properties that hold a.a.s.² in a uniformly random $\mathbf{STS}(n)$. This turns out to be surprisingly difficult: despite the enormous advances in the theory of random combinatorial structures since the foundational work of Erdős and Rényi [10], almost none of the available tools seem to be applicable to random Steiner triple systems. Random Steiner triple systems lack independence or any kind of recursive structure, which rules out many of the techniques used to study Erdős–Rényi random graphs and random permutations, and there is basically no freedom to make local changes, which precludes the use of “switching” techniques often used in the study of random regular graphs (see for example [29]). It is not even clear how to study a random $\mathbf{STS}(n)$ empirically: in an attempt to find an efficient algorithm to generate a random $\mathbf{STS}(n)$, Cameron [5] designed a Markov chain on Steiner triple systems, but he was not able to determine whether this chain was connected. As far as we know, before the work of Kwan [31], the only nontrivial fact known to hold for a random $\mathbf{STS}(n)$ was that it a.a.s. has trivial automorphism group, a fact proved by Babai [2] using a direct (and rather coarse) counting argument.

Building on the ideas in [31], introducing the sparse regularity method and extending a random partitioning argument from [11], in this paper we prove the following “asymptotic” version of Conjecture 1.1, adding to the short list of known facts about random Steiner triple systems.

Theorem 1.2. *Let \mathcal{S} be a uniformly random $\mathbf{STS}(n)$, where $n \equiv 3 \pmod{6}$. Then a.a.s. \mathcal{S} has $(1/2 - o(1))n$ disjoint perfect matchings.*

Our proof of Theorem 1.2 is based on tools and intuition from random *hypergraph* theory. We conclude this introduction with a brief discussion of random hypergraphs and some related work.

For $k \geq 2$, a k -uniform hypergraph H (or a k -graph for short) is a pair $H = (V, E)$, where V is a finite set of *vertices* and $E \subseteq \binom{V}{k}$ is a family of k -element subsets of V , referred to as *edges*. The existence of perfect matchings is one of the most central questions in the theory of graphs and hypergraphs. In the case of graphs (that is, 2-uniform hypergraphs), the problem of finding a perfect matching (if one exists) is relatively simple, but the analogous problem in the hypergraph setting is known to be NP-hard (see [18]). Therefore, a main theme in extremal and probabilistic combinatorics is to investigate sufficient conditions for the existence of perfect matchings in hypergraphs. The results which are most relevant to our paper are those regarding the problem of finding perfect matchings in *random* hypergraphs.

Let $\mathbf{H}^k(n, p)$ be the (binomial) random k -uniform hypergraph distribution on the vertex set $V(H) = [n]$, where each k -set $X \in \binom{[n]}{k}$ is included as an edge with probability p independently. A main problem in this area was to find a “threshold” function $q(n)$ for the property of containing a perfect matching. That is, to find a function q for which, when $\mathbf{G} \sim \mathbf{H}^k(n, p)$, we have

$$\Pr[\mathbf{G} \text{ contains a perfect matching}] \rightarrow \begin{cases} 1 & \text{if } p(n)/q(n) \rightarrow \infty \\ 0 & \text{if } p(n)/q(n) \rightarrow 0. \end{cases}$$

Following a long line of work, the most significant development in this area is the celebrated work of Johansson, Kahn and Vu [17] where they showed that $q(n) = \log n/n$ defines a threshold function

²By “asymptotically almost surely”, or “a.a.s.”, we mean that the probability of an event is $1 - o(1)$. Here and for the rest of the paper, asymptotics are as $n \rightarrow \infty$.

for this property. Two other relevant results are those of Frieze and Krivelevich [13] and its variant due to Ferber, Kronenberg and Long [11] which establish a way to find “many” edge-disjoint perfect matchings/Hamiltonian cycles in random (hyper)graphs based on some pseudorandom properties and a random partitioning argument.

1.1 Structure of the paper

The structure of the paper is as follows. In Section 2 we review the methods introduced in [31] for studying random Steiner triple systems via the triangle removal process, and present one new lemma for studying monotone decreasing properties. In Section 3 we outline the general approach of the proof, stating some properties that we will use to pack perfect matchings and that we will be able to show hold a.a.s. in a random Steiner triple system.

Before turning to the proof of Theorem 1.2, in Section 4 we discuss the sparse regularity lemma for hypergraphs and some auxiliary lemmas for applying it, and in Section 5 we discuss some lemmas for finding almost-perfect matchings in hypergraphs.

The proof of Theorem 1.2 itself appears in Sections 6 and 7. In Section 6 we prove that certain properties suffice for packing perfect matchings, and in Section 7 we show that these properties a.a.s. hold in random Steiner triple systems.

Finally, in Section 8 we have some concluding remarks, and in Appendix A we explain how to generalise Conlon, Gowers, Samotij and Schacht’s proof [8] of the so-called KŁR conjecture to “linear” hypergraphs. This will be a key tool in our proof.

1.2 Notation

We use standard asymptotic notation throughout, as follows. For functions $f = f(n)$ and $g = g(n)$, we write $f = O(g)$ to mean that there is a constant C such that $|f| \leq C|g|$, $f = \Omega(g)$ to mean that there is a constant $c > 0$ such that $f(n) \geq c|g(n)|$ for sufficiently large n , $f = \Theta(g)$ to mean that that $f = O(g)$ and $f = \Omega(g)$, and $f = o(g)$ to mean that $f/g \rightarrow 0$ as $n \rightarrow \infty$. Also, following [20], the notation $f = 1 \pm \varepsilon$ means $1 - \varepsilon \leq f \leq 1 + \varepsilon$.

We also use standard graph theory notation: $V(G)$ and $E(G)$ are the sets of vertices and (hyper)edges of a (hyper)graph G , and $v(G)$ and $e(G)$ are the cardinalities of these sets. The subgraph of G induced by a vertex subset U is denoted $G[U]$, the degree of a vertex v is denoted $\deg_G(v)$, and the subgraph obtained by deleting v is denoted $G - v$. Given a subset of vertices $U \subseteq V(G)$ and any vertex $v \in V(G)$, we let $\deg_U(v)$ denote the degree of v into the subset U (that is, the number of edges consisting of v and some vertices of U).

For a positive integer n , we write $[n]$ for the set $\{1, 2, \dots, n\}$. For a real number x , the floor and ceiling functions are denoted $\lfloor x \rfloor = \max\{i \in \mathbb{Z} : i \leq x\}$ and $\lceil x \rceil = \min\{i \in \mathbb{Z} : i \geq x\}$. We will however mostly omit floor and ceiling signs and assume large numbers are integers, wherever divisibility considerations are not important. We will use the convention that random objects (for example, random variables or random graphs) are printed in bold. Finally, all logarithms are in base e .

2 Random Steiner triple systems via the triangle removal process

In this section we reproduce the general theorems from [31] for studying the behaviour of a randomly chosen **STS**(n) via the triangle removal process, including a new lemma for studying monotone decreasing properties. This machinery will be crucial to prove Theorem 1.2.

Note that any **STS**(n) is a 3-graph and let $N = \binom{n}{2}/3 = (1 + o(1))n^2/6$ be the number of edges in any **STS**(n). We assume throughout this section that n is 1 or 3 mod 6 (as otherwise an **STS**(n) does not exist). Let us first make some useful definitions.

Definition 2.1 (partial Steiner triple systems). A *partial Steiner triple system* (also known as a *linear 3-graph*) is a 3-graph on the vertex set $[n]$ in which every pair of vertices is included in at most one edge. We will also want to consider partial Steiner triple systems equipped with an ordering on their edges.

Let \mathcal{O} be the set of ordered Steiner triple systems, and let \mathcal{O}_m be the set of ordered partial Steiner triple systems with exactly m edges. For $S \in \mathcal{O}_m$ and $i \leq m$, let S_i be the ordered partial Steiner triple system consisting of just the first i edges of S . For a (possibly ordered) partial Steiner triple system S , let $G(S)$ be the graph (that is, a 2-graph) with an edge for every pair of vertices which does not appear in any edge of S . So, if S has m edges, then $G(S)$ has $\binom{n}{2} - 3m$ edges.

Definition 2.2 (quasirandomness). For a graph G with n vertices and m edges, let $d(G) = m/\binom{n}{2}$ denote its density. We say G is (ε, h) -quasirandom if for every set A of at most h vertices, we have $|\bigcap_{w \in A} N_G(w)| = (1 \pm \varepsilon)d(G)^{|A|}n$. Let $\mathcal{O}_m^{\varepsilon, h} \subseteq \mathcal{O}_m$ be the set of ordered partial Steiner triple systems $S \in \mathcal{O}_m$ such that $G(S_i)$ is (ε, h) -quasirandom for each $i \leq m$.

Definition 2.3 (the triangle removal process). The triangle removal process is defined as follows. Start with the complete graph K_n and iteratively delete the edge-set of a triangle chosen uniformly at random from all triangles in the remaining graph. If we continue this process for m steps, the deleted triangles (in order) can be interpreted as an ordered partial Steiner triple system in \mathcal{O}_m . It is also possible that the process aborts (because there are no triangles left) before m steps, in which case we say it returns the value “*”. We denote by $R(n, m)$ the resulting distribution on $\mathcal{O}_m \cup \{*\}$.

Now, we can state the general theorem from [31] comparing a typical **STS**(n) with a typical outcome of the triangle removal process. Basically, if we can show that the first few edges of the triangle removal process (as an ordered partial Steiner triple system) satisfy some property with extremely high probability, then it follows that the first few edges of a uniformly random ordered **STS**(n) satisfy the same property with high probability. Moreover, it suffices to study the triangle removal process conditioned on some “good” event, provided that this event contains the event that (the graph of uncovered edges of) our partial Steiner triple system is sufficiently quasirandom.

Theorem 2.4. Fixing $h \in \mathbb{N}$ and sufficiently small $a > 0$, there is $b = b(a, h) > 0$ such that the following holds. Fix $\alpha \in (0, 1)$, let $\mathcal{P} \subseteq \mathcal{O}_{\alpha N}$ be a property of ordered partial Steiner triple systems, let $\varepsilon = n^{-a}$, let $\mathcal{Q} \supseteq \mathcal{O}_{\alpha N}^{\varepsilon, h}$, let $S \in \mathcal{O}$ be a uniformly random ordered Steiner triple system and let $S' \sim R(n, \alpha N)$. If

$$\Pr(S' \notin \mathcal{P} \mid S' \in \mathcal{Q}) \leq \exp(-n^{2-b})$$

then

$$\Pr(S_{\alpha N} \notin \mathcal{P}) \leq \exp(-\Omega(n^{1-2a})).$$

2.1 A coupling lemma

The triangle removal process can be rather technical to study directly, so [31] included a general lemma approximating the first few steps of the triangle removal process by a binomial random 3-graph with relatively small edge probability. The idea is that instead of randomly choosing triples one-by-one avoiding conflicts with previous choices, one can randomly choose several triples at once, and just delete those triples which conflict with each other. If the edge probability is small, there are likely to be few conflicts, so these two processes (at least intuitively) give almost the same distribution.

This lemma in [31] (specifically, [31, Lemma 2.10]) was suitable for studying properties \mathcal{P} that are monotone increasing in the sense that $S \in \mathcal{P}$ and $S' \supseteq S$ implies $S' \in \mathcal{P}$. Surprisingly, despite the fact that the existence of perfect matchings (or collections of disjoint perfect matchings) is a monotone increasing property, in this paper we will instead need a similar lemma for monotone decreasing properties. Before stating the lemma we need the following definition.

Definition 2.5. For a partial Steiner triple system S , let $R(S, m)$ be the partial Steiner triple system distribution obtained with m steps of the triangle removal process starting from $G(S)$. (So, if S has m' edges, then $R(S, m)$ has $m + m'$ edges, unless the triangle removal process aborts).

Lemma 2.6. Fix sufficiently small $\alpha \in (0, 1)$. Consider some $S \in \mathcal{O}_m^{\alpha, 2}$ for some $m \leq \alpha N$, and consider a property \mathcal{P} of unordered 3-graphs (which may depend on S) that is monotone decreasing in the sense that $G \in \mathcal{P}$ and $G' \subseteq G$ implies $G' \in \mathcal{P}$. Let $\mathbf{S} \sim R(S, \alpha N)$ and $\mathbf{G} \sim \mathbf{H}^3(n, q)$ with $q = \alpha(1 + 10\alpha)/n$. Then

$$\Pr(\mathbf{S} \notin \mathcal{P}) \leq \Pr(\mathbf{G} \notin \mathcal{P}) + e^{-\Omega(n^2)}.$$

The proof of Lemma 2.6 is somewhat more complicated than [31, Lemma 2.10], but the intuition is basically the same: if α is small and S is a partial Steiner triple system with few edges, then the outcome of $H^3(n, q)$ is likely to “almost” be a partial Steiner triple system and “almost” avoid conflicts with S , and therefore approximates $R(S, \alpha N)$. For the proof we will need the following concentration inequality, which appears as [31, Theorem 2.11], and is also a direct consequence of [45, Theorem 1.3]. It is a bounded-differences inequality with Bernstein-type tails which can be used to analyse sparse random hypergraphs. Standard bounded-difference inequalities such as the Azuma–Hoeffding inequality do not provide strong enough tail bounds to apply Theorem 2.4.

Theorem 2.7. *Let $\omega = (\omega_1, \dots, \omega_n)$ be a sequence of independent, identically distributed random variables with $\Pr(\omega_i = 1) = p$ and $\Pr(\omega_i = 0) = 1 - p$. Let $f : \{0, 1\}^n \rightarrow \mathbb{R}$ satisfy the Lipschitz condition $|f(\omega) - f(\omega')| \leq K$ for all pairs $\omega, \omega' \in \{0, 1\}^n$ differing in exactly one coordinate. Then*

$$\Pr(|f(\omega) - \mathbb{E}f(\omega)| > t) \leq \exp\left(-\frac{t^2}{4K^2np + 2Kt}\right).$$

We will also need the fact that quasirandom graphs have approximately the “right” number of triangles.

Lemma 2.8. *Let ε be sufficiently small and let G be an $(\varepsilon, 2)$ -quasirandom graph with density d . Then G has $(1 \pm 3\varepsilon)d^3n^3/6$ triangles.*

Proof. By $(\varepsilon, 1)$ -quasirandomness, the sum of the degrees is $(1 \pm \varepsilon)dn^2$, so there are $(1 \pm \varepsilon)dn^2/2$ edges. For each such edge, by $(\varepsilon, 2)$ -quasirandomness there are $(1 \pm \varepsilon)d^2n$ common neighbours of that edge, each of which gives a triangle containing that edge. Therefore the total number of triangles is

$$((1 \pm \varepsilon)dn^2/2)((1 \pm \varepsilon)d^2n)/3 = (1 \pm 3\varepsilon)d^3n^3/6$$

as desired. □

Now we can prove Lemma 2.6.

Proof of Lemma 2.6. Let \mathbf{S}^* be obtained from \mathbf{G} by deleting every edge which intersects an edge of S or another edge of \mathbf{G} in two vertices. We can couple $(\mathbf{S}^*, \mathbf{G})$ and \mathbf{S} in such a way that $\mathbf{G} \supseteq \mathbf{S}$ as long as \mathbf{S}^* has at least αN edges (randomly order the edges in \mathbf{G} and run the triangle removal process with this ordering). Let \mathbf{Y} be the number of edges in \mathbf{S}^* , which is the number of edges in \mathbf{G} that do not conflict with S and are *isolated* in the sense that they do not intersect any other edge of \mathbf{G} in more than one vertex. By Lemma 2.8, there are at least $(1 \pm 3\alpha)(1 - \alpha)^3n^3/6 = (1 \pm 7\alpha)n^3/6$ possibilities for such an edge, and each is present and isolated with probability

$$(\alpha(1 + 10\alpha)/n)(1 - \alpha(1 + 10\alpha)/n)^{3(n-3)+1} = \alpha(1 + 10\alpha - o(1))e^{-3\alpha(1+10\alpha)}/n = (1 + 7\alpha + O(\alpha^2))\alpha/n$$

for sufficiently small α . This implies that $\mathbb{E}\mathbf{Y} = \alpha N + \Omega(n^2)$. Next, observe that adding an edge to \mathbf{G} can increase \mathbf{Y} by at most 1, and removing an edge to \mathbf{G} can increase \mathbf{Y} by at most 3 (by making three edges isolated). So, by Theorem 2.7,

$$\Pr(\mathbf{Y} < \alpha N) \leq \Pr(|\mathbf{Y} - \mathbb{E}\mathbf{Y}| \geq \Omega(n^2)) \leq \exp\left(-\Omega\left(\frac{(n^2)^2}{3^2\binom{n}{3}(\alpha(1 + 10\alpha)/n) + 3n^2}\right)\right) = e^{-\Omega(n^2)}.$$

It follows that

$$\Pr(\mathbf{S} \notin \mathcal{P}) \leq \Pr(\mathbf{G} \notin \mathcal{P}) + \Pr(\mathbf{Y} < \alpha N) = \Pr(\mathbf{G} \notin \mathcal{P}) + e^{-\Omega(n^2)}. \quad \square$$

3 Sufficient properties for packing

In this section we state some lemmas from which Theorem 1.2 will follow. Basically, the idea is to define some properties which can be shown to hold a.a.s. in random Steiner triple systems, and which can be used to find “many” perfect matchings. The first of these properties is an “upper-quasirandomness” condition, which is a requirement to effectively apply the so-called regularity method in the sparse setting.

Definition 3.1. For vertex sets X, Y, Z in a 3-graph G , let $e_G(X, Y, Z)$ be the number of orderings (x, y, z) of edges $\{x, y, z\}$ with $x \in X, y \in Y, z \in Z$ (if X, Y, Z are disjoint this is the number of edges with exactly one vertex in each of X, Y, Z). A 3-graph is (p, β) -upper-quasirandom if for any vertex subsets X, Y, Z , we have $e(X, Y, Z) \leq p|X||Y||Z| + \beta n^3 p$.

The second property we will need is the existence of certain special subgraphs which we refer to as *absorbers*. We will later see that it is not very hard to construct “almost”-perfect matchings, and we will be able to use the special features of absorbers to complete almost-perfect matchings into perfect matchings (by “absorbing” the uncovered vertices). This idea falls into the framework of the *absorption method*, which was first introduced as a general method by Rödl, Ruciński and Szemerédi in [41], and has had numerous applications since.

Definition 3.2 (sub-absorbers and absorbers). The absorbers we use (see also Figure 1 for an illustration) are based on *sub-absorbers* and will be defined in two steps:

- A *sub-absorber* rooted on a triple of vertices x, y, z is a set of five edges of the form

$$\{\{x, x_1, x_2\}, \{y, y_1, y_2\}, \{z, z_1, z_2\}, \{x_1, y_1, z_1\}, \{x_2, y_2, z_2\}\}.$$

We call x, y, z the *rooted vertices* of the sub-absorber and we call the other nine vertices the *external vertices*. If an edge contains a rooted vertex, we call it a *rooted edge*.

- An *absorber* rooted on a triple of vertices x, y, z is a set of 13 edges obtained in the following way. Put three disjoint edges $\{x, x_1, x_2\}, \{y, y_1, y_2\}, \{z, z_1, z_2\}$, then put a sub-absorber rooted on $\{x_1, y_1, z_1\}$ and a sub-absorber rooted on $\{x_2, y_2, z_2\}$ (in such a way that no pair of edges intersects in more than one vertex). We call x, y, z the *rooted vertices* of the absorber and the other 18 vertices the *external vertices*. If an edge contains a rooted vertex, we call it a *rooted edge*. Note that the edges of an absorber can be partitioned into two matchings: a perfect matching with seven edges (which we call the *covering matching*), and a matching with six edges (which we call the *non-covering matching*).

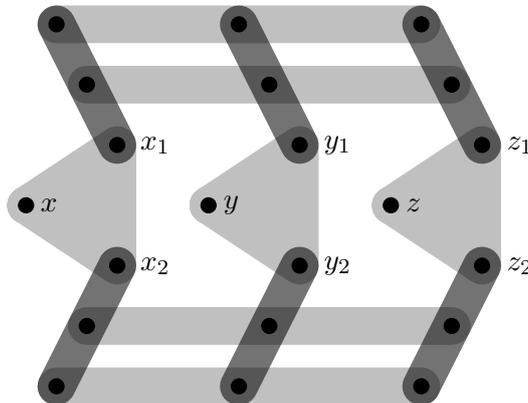


Figure 1. An illustration of an absorber for (x, y, z) . The light edges are the covering matching and the dark edges are the non-covering matching.

The crucial property of an absorber, that allows it to be used to complete a matching into a perfect matching, is the existence of the covering and non-covering matching (we can choose whether to cover the root vertices x, y, z or not). The observant reader may notice that sub-absorbers themselves are simpler structures also having a covering and non-covering matching; the reason we consider the larger absorbers is that it is easier to find a rooted absorber in a random 3-graph than a rooted sub-absorber (mainly due to the fact that the rooted edges in an absorber have no common incident edges).

In the proof of our main result we will want to construct our perfect matchings one-by-one, so it will be important that we can find absorbers disjoint to a set of perfect matchings that have already been constructed. In order to do so, we will show that the 3-graphs we are working with are “resilient” with respect to absorbers in the sense that every “dense” subgraph contains at least one absorber for each triple.

Definition 3.3. A 3-graph G is D -absorber-resilient if every subgraph $G' \subseteq G$ with minimum degree at least D has an absorber rooted on every triple x, y, z of distinct vertices of G .

Now, the properties we will use for packing perfect matchings are as follows. It will be useful for the reader to think of β as a very small constant which controls the error terms, and α as a constant that is much larger (but still quite small in absolute terms) that measures the fraction of a Steiner triple system we are considering.

Definition 3.4. Consider $\alpha, \beta > 0$, and let S be any 3-graph. We say that S is (α, β) -good if it satisfies the following properties.

1. **Almost-regularity:** for every vertex v we have $\deg_S(v) = \alpha n/2 \pm \beta n$;
2. **Pseudorandomness:** S is $(\alpha/n, \beta)$ -upper-quasirandom;
3. **Robust absorber-resilience:** for each $w \geq \beta n$, letting $\eta = w/n$, all but at most

$$\exp(-10^{-8}(w/n)^4 \alpha^2 n) \binom{n}{w}$$

of the w -vertex induced subgraphs $S[W]$ are $0.999\eta^2\alpha(n/2)$ -absorber-resilient.

Our proof strategy for finding many perfect matchings will roughly go as follows. Suppose that \mathcal{S} is a typical $\mathbf{STS}(n)$. We show that (the edge-set of) \mathcal{S} can be decomposed into (α, β) -good subgraphs, for some carefully chosen α, β . Then, in each of these subgraphs, we show that one can approximately decompose its edges into perfect matchings. Together, we obtain our desired collection of $(1 - o(1))n/2$ perfect matchings.

One might wonder why we need this intermediate stage of partitioning \mathcal{S} , instead of working with \mathcal{S} directly. The reason we do so is that “small bits” of \mathcal{S} behave like the binomial random hypergraph model (recall Lemma 2.6) so we can borrow some tools and ideas from this well-studied model in order to construct our matchings.

Now we state our key lemmas, which together imply Theorem 1.2.

Lemma 3.5. *There is $r \in \mathbb{N}$ such that the following holds. Suppose n is congruent to 1 or 3 mod 6 and let \mathcal{S} be a uniformly random $\mathbf{STS}(n)$. Then a.a.s. the edge-set of \mathcal{S} can be partitioned into r spanning subgraphs that are $(1/r, o(1))$ -good.*

Lemma 3.6. *For any $\alpha = \Omega(1)$ and $n \equiv 0 \pmod{3}$, every n -vertex $(\alpha, o(1))$ -good linear 3-graph S (that is, a partial Steiner triple system) has $(\alpha/2 - o(1))n$ disjoint perfect matchings.*

The proof of Lemma 3.5 mainly comes down to studying the robust absorber-resilience property of goodness. (The almost regularity and pseudorandomness properties can be proved in a fairly straightforward fashion using Theorem 2.4 and Lemma 2.6). For the robust absorber-resilience property, we will use Theorem 2.4 and Lemma 2.6 in combination with a hypergraph generalisation of the sparse regularity lemma (to be stated in Section 4) and a hypergraph generalisation of the resolution of the KLR conjecture by Conlon, Gowers, Samotij and Schacht (to be stated in Section 4 as well). Roughly speaking, this allows us to reduce certain problems about subgraphs of a random hypergraph to problems about *dense* hypergraphs, and the latter case is much more tractable to study. One complication is that even though robust absorber-resilience is a property of subgraphs of a random hypergraph, an absorber is a *rooted* structure. We will need some additional tricks to reduce the situation to one where the KLR conjecture can actually be applied. The details of the proof will be presented in Section 7.

Concerning Lemma 3.6, one may wonder how the robust absorber-resilience property of goodness could be strong enough to produce an approximate decomposition of \mathcal{S} into perfect matchings. Naïvely, it seems that after we have found only $0.001\alpha n/2$ perfect matchings, this property no longer gives us any guarantee for the existence of absorbers in the remaining edges. Indeed, the proof of Lemma 3.6 would be much simpler if the robust absorber-resilience property of goodness could be strengthened to guarantee that \mathcal{S} is D -absorber-resilient for $D = o(n)$. Unfortunately, it is not clear how to prove that such a strong property holds in a typical $\mathbf{STS}(n)$, even with the KLR conjecture in hand. Instead, we employ a random partitioning trick inspired by the work of Ferber, Kronenberg and Long [11]. We partition the edges of

\mathcal{S} into many subgraphs with different roles, some of which are used to find almost-perfect matchings and some of which are used to complete almost-perfect matchings into perfect matchings (using absorbers). The latter subgraphs contain only a small fraction of the edges of \mathcal{S} , but have comparatively high degree (this is possible because each of these subgraphs have a very small number of vertices). We then only need a weak absorber-resilience property for these smaller subgraphs. The details of the proof will be presented in Section 6.

4 Sparse regularity and the KLR conjecture for hypergraphs

Kohayakawa and Rödl [27] proved a sparse version of the so-called regularity lemma for graphs. In this paper, for the proofs of Lemmas 3.5 and 3.6 we will need a generalisation of the sparse regularity lemma to hypergraphs. Fortunately, while the general hypergraph regularity lemma is much more complicated to prove (and state) than the graph regularity lemma, for our purposes (embedding *linear* hypergraphs) we only need a sparse version of the “weak” hypergraph regularity lemma (see for example [28, Theorem 9]). To state our sparse regularity lemma for hypergraphs we need a few definitions.

Definition 4.1. Let $\varepsilon, \eta > 0$ and $0 \leq p \leq 1$ be arbitrary parameters.

- **Density:** Consider disjoint vertex sets X_1, \dots, X_r in an r -graph G . Let $e(X_1, \dots, X_r)$ be the number of edges with a vertex in each X_i , and let

$$d(X_1, \dots, X_r) = \frac{e(X_1, \dots, X_r)}{|X_1| \dots |X_r|}$$

be the *density* between X_1, \dots, X_r .

- **Regular tuples:** An r -graph G is said to be *r -partite* if its vertex set $V(G)$ consists of a partition $V(G) = V_1 \cup \dots \cup V_r$ into r parts in such a way that each of its edges intersect each V_i in exactly one vertex. An r -partite r -graph with parts V_1, \dots, V_r is (ε, p) -*regular* if, for every $V'_1 \subseteq V_1, \dots, V'_r \subseteq V_r$ with $|V'_i| \geq \varepsilon|V_i|$, the density $d(V'_1, \dots, V'_r)$ of edges between V'_1, \dots, V'_r satisfies

$$|d(V'_1, \dots, V'_r) - d(V_1, \dots, V_r)| \leq \varepsilon p.$$

- **Regular partitions:** A partition of the vertex set of a r -graph into t parts V_1, \dots, V_t is said to be (ε, p) -regular if it is an equipartition, and for all but at most εt^r r -tuples $(V_{i_1}, \dots, V_{i_r})$, the induced r -partite r -graph between V_{i_1}, \dots, V_{i_r} is (ε, p) -regular.
- **Upper-uniformity:** An r -graph G is (η, p, D) -*upper-uniform* if for any choice of disjoint subsets U_1, \dots, U_r with $|U_1|, \dots, |U_r| \geq \eta|V(G)|$, we have $d(U_1, \dots, U_r) \leq Dp$.

Note that upper-uniformity is a weaker property than upper-quasirandomness: if a 3-graph G is $(o(1), p)$ -upper-quasirandom, then it is $(o(1), p, 1 + o(1))$ -upper-uniform. Now, our hypergraph regularity lemma is as follows. We omit its proof since it is straightforward to adapt a proof of the sparse graph regularity lemma (see [27] for a sparse regularity lemma for graphs, and see [28, Theorem 9] for a weak regularity lemma for dense hypergraphs).

Lemma 4.2. For every $\varepsilon, D > 0$ and every positive integer t_0 , there exist $\eta > 0$ and $T \in \mathbb{N}$ such that for every $p \in [0, 1]$, every (η, p, D) -upper-uniform r -graph G with at least t_0 vertices admits an (ε, p) -regular partition V_1, \dots, V_t of its vertex set into $t_0 \leq t \leq T$ parts.

We will use Lemma 4.2 for several different purposes. Crucial to all of these is the notion of a *cluster hypergraph* which is a dense hypergraph which encodes the large-scale structure of a regular partition. From now on we return to considering only 3-graphs.

Definition 4.3. Given an (ε, p) -regular partition V_1, \dots, V_t of the vertex set of a 3-graph G , the *cluster hypergraph* is the 3-graph whose vertices are the clusters V_1, \dots, V_t , with an edge $\{V_i, V_j, V_k\}$ if $d(V_i, V_j, V_k) > 2\varepsilon p$ and the induced tripartite 3-graph between V_i, V_j and V_k is (ε, p) -regular.

If the regularity lemma is applied with small ε and large t_0 , the cluster hypergraph approximately inherits certain density properties from the original graph G , as follows.

Lemma 4.4. *Fix sufficiently small $\varepsilon > 0$ and sufficiently large $t_0 \in \mathbb{N}$, and let G be an n -vertex $(p, o(1))$ -upper-quasirandom 3-graph. Let $G' \subseteq G$ be a spanning subgraph with minimum degree at least $\delta p \binom{n}{2}$. Let \mathcal{R} be the t -vertex cluster 3-graph obtained by applying the sparse regularity lemma to G' with parameters t_0, p and ε . Then all but $\sqrt{\varepsilon}t$ vertices of \mathcal{R} have degrees at least $(\delta - 3\sqrt{\varepsilon} - 3/t_0)\binom{t}{2}$.*

The proof of Lemma 4.4 is by a standard counting argument. It is a special case of Lemma 4.8, to appear in the next subsection (so we defer the proof until then).

An immediate application of Lemma 4.4 is the fact that high-degree subgraphs of upper-quasirandom 3-graphs have a ‘‘rich’’ vertex subset Z such that most vertices outside Z have reasonably high degree into Z . This will be important for the proof of Lemma 3.6.

Lemma 4.5. *For any $\delta > 0$ and any $\varepsilon > 0$ that is sufficiently small relative to δ , there is $\xi > 0$ such that the following holds. Let G be an $(p, o(1))$ -upper-quasirandom 3-graph. Let $G' \subseteq G$ be a spanning subgraph with minimum degree $\delta p \binom{n}{2}$. Then there is a $2\varepsilon n$ -vertex subset $U \subseteq V(G)$ such that the following conditions hold.*

1. *For all but at most $2\sqrt{\varepsilon}n$ vertices v , there are at least $\xi \binom{n}{2} p$ edges of G' containing v and two vertices of U ;*
2. *every subset of $|U| - \xi n$ vertices of U induces at least one edge of G' .*

Proof. Apply the sparse regularity lemma (Lemma 4.2) to G' with small ε and large t_0 , and let U contain a 2ε -fraction of each cluster. Let \mathcal{R} be the cluster 3-graph (with $t \leq T$ vertices).

By Lemma 4.4, if ε is small enough and t_0 is large enough, all but $\sqrt{\varepsilon}t$ clusters V_i have degree at least $(\delta/2)\binom{t}{2}$ in the cluster graph, and in particular participate in an (ε, p) -regular triple V_i, V_j, V_q with $d(V_i, V_j, V_q) > 2\varepsilon p$ (recall that we are assuming that ε is small relative to δ).

Now, take $\xi = \varepsilon(1/T^2)$. To verify the first part of Lemma 4.5, we observe that for any cluster V_i as above (participating in an (ε, p) -regular triple V_i, V_j, V_q with $d(V_i, V_j, V_q) > 2\varepsilon p$), at least a $(1 - \varepsilon)$ -fraction of the vertices $v \in V_i$ satisfy the condition in the first part of Lemma 4.5. Indeed, let $V'_j = V_j \cap U$ and $V'_q = V_q \cap U$, and let V'_i be the set of vertices $v \in V_i$ for which there are fewer than $\varepsilon|V'_j||V'_q|p$ edges containing v , a vertex in V'_j and a vertex in V'_q . Then, $d(V'_i, V'_j, V'_q) < \varepsilon p$, and if we had $|V'_i| \geq \varepsilon|V_i|$ this would contradict (ε, p) -regularity. So, $|V'_i| < \varepsilon|V_i|$. Then, observe that $\varepsilon|V'_j||V'_q|p \geq \xi \binom{n}{2} p$.

For the second property, if we delete fewer than ξn vertices from U then we still have an ε -fraction of each cluster V_i and therefore have at least one edge. \square

4.1 Refining an existing partition

In our proof of Lemma 3.5 we will apply the sparse regularity lemma to a 3-graph whose vertices are already partitioned into a few different parts with different roles. It will be important that the regular partition in Lemma 4.2 can be chosen to be consistent with this existing partition.

Lemma 4.6. *Suppose that a 3-graph G has its vertices partitioned into sets U_1, \dots, U_h . In the (ε, p) -regular partition guaranteed by Lemma 4.2, we can assume that all but at most εht of the clusters V_i are contained in some U_j .*

For the reader who is familiar with the proof of the regularity lemma, the proof of Lemma 4.6 is straightforward. Indeed, in order to prove the regularity lemma, one starts with an arbitrary partition and keeps refining it in a clever way. Nevertheless, for completeness we include a short reduction from Lemma 4.2.

Proof of Lemma 4.6. Apply Lemma 4.2 with regularity parameter $\varepsilon^2/2$. For each cluster V_i , order the vertices in V_i according to the partition U_1, \dots, U_h (first the vertices from U_1 , then from U_2 , etc). Now, equipartition V_i into $\lceil 1/\varepsilon \rceil$ intervals $V_i^1, \dots, V_i^{\lceil 1/\varepsilon \rceil}$ with respect to this ordering. At most h of these intervals intersect multiple U_j , and the V_i^q form an (ε, p) -regular partition. (Strictly speaking some of the clusters may now have sizes differing by 2 instead of 1, but we can move some vertices between clusters to correct this without having any material effect on the regularity of the partition). \square

We also need a more technical version of Lemma 4.4 translating the degrees between the U_i into degrees in the cluster graph. First, we generalise the definition of a cluster graph.

Definition 4.7. Given a 3-graph G with vertex partition U_1, \dots, U_h and an (ε, p) -regular partition V_1, \dots, V_t of its vertices, the *partitioned cluster graph* \mathcal{R} with *threshold* τ is the 3-graph defined as follows. The vertices of \mathcal{R} are the clusters V_i which are completely contained in some U_j , with an edge $\{V_i, V_j, V_k\}$ if $d(V_i, V_j, V_k) > \tau p$ and the induced tripartite 3-graph between V_i, V_j and V_k is (ε, p) -regular.

Lemma 4.8. *Fix sufficiently small $\varepsilon > 0$ and sufficiently large $t_0 \in \mathbb{N}$, and let G be an n -vertex $(p, o(1))$ -upper-quasirandom 3-graph with a partition U_1, \dots, U_h of its vertices into parts of sizes u_1, \dots, u_h , respectively. Let $G' \subseteq G$ be a spanning subgraph such that each $v \in U_i$ has $\deg_{U_j}(v) \geq \delta_{ij} p \binom{u_j}{2}$ (where the degree here is with respect to G').*

Let \mathcal{R} be the partitioned cluster 3-graph with threshold τ obtained by applying Lemmas 4.2 and 4.6 to G' with parameters t_0, p and ε . Let t_i be the number of clusters contained in U_i and let \mathcal{U}_i be the set of clusters contained in U_i . Then each $t_i \geq (u_i/n)t - \varepsilon ht$ and for each i, j , all but $\sqrt{\varepsilon}t$ clusters $X \in \mathcal{U}_i$ have $\deg_{\mathcal{U}_j}(X) \geq \delta_{ij} \binom{t_i}{2} - t^2(\tau + 2\varepsilon h + \sqrt{\varepsilon} + 2/t_0)$ in the cluster graph \mathcal{R} .

Note that Lemma 4.4 is actually a special case of Lemma 4.8 (taking the trivial partition and threshold 2ε).

Proof of Lemma 4.8. The clusters in \mathcal{U}_i comprise at most $t_i(n/t)$ vertices, so $|\mathcal{U}_i| = u_i \leq t_i(n/t) + \varepsilon ht(n/t)$. It follows that $t_i \geq (u_i/n)t - \varepsilon ht$ as claimed.

Let \mathcal{W} be the set of all clusters W in the (ε, p) -regular partition for which there are more than $\sqrt{\varepsilon} \binom{t}{2}$ non- (ε, p) -regular triples (V_i, V_j, W) containing W . There can be at most $3\varepsilon \binom{t}{3} / (\sqrt{\varepsilon} \binom{t}{2}) \leq \sqrt{\varepsilon}t$ clusters in \mathcal{W} . Let Z be the set of at most εhn vertices whose cluster does not appear in the cluster 3-graph (because the cluster was not completely contained in any U_i).

If a cluster $X \in \mathcal{U}_i \setminus \mathcal{W}$ has $\deg_{\mathcal{U}_j}(X) = d$ in the cluster 3-graph, then by $(p, o(1))$ -upper-quasirandomness the number of edges of G' with a vertex in X and two vertices in U_j is at most

$$\begin{aligned} & d(1 + o(1))p \binom{n}{t}^3 + \sqrt{\varepsilon} \binom{t}{2} p \binom{n}{t}^3 + \tau \binom{t_j}{2} p \binom{n}{t}^3 + e_G(X, Z \cup X, U_j) + \sum_{W \in \mathcal{V}(\mathcal{R})} e_G(X, W, W) \\ & \leq (1 + o(1))p \binom{n}{t}^3 (d + t^2(\tau + \sqrt{\varepsilon}) + t(\varepsilon ht + 1) + t). \end{aligned}$$

But by the degree assumption in G' and the fact that $u_j \geq t_j(n/t)$ this number is at least

$$(n/t)\delta_{ij}p \binom{u_j}{2} \geq p\delta_{ij} \binom{t_j}{2} \binom{n}{t}^3.$$

It follows that

$$d \geq \delta_{ij} \binom{t_j}{2} - t^2(\tau + 2\varepsilon h + \sqrt{\varepsilon} + 2/t).$$

□

4.2 The KLR conjecture

One of the most powerful aspects of the sparse regularity method is that, for a subgraph G of a typical outcome of a random graph, if we find a substructure in the cluster graph (which is usually dense, therefore comparatively easy to analyse), then a corresponding structure must also exist in the original graph G . For graphs, this was conjectured to be true by Kohayakawa, Łuczak and Rödl [26], and was proved by Conlon, Gowers, Samotij and Schacht [8]. We will need a generalisation to hypergraphs, which again we state in more general form than we need, in case it is useful for future applications. First we need some definitions.

Definition 4.9. Consider an r -graph H with vertex set $\{1, \dots, k\}$ and let $\mathcal{G}(H, n, m, p, \varepsilon)$ be the collection of all r -graphs G obtained in the following way. The vertex set of G is a disjoint union $V_1 \cup \dots \cup V_k$ of sets of size n . For each edge $\{i_1, \dots, i_r\} \in E(H)$, we add to G an (ε, p) -regular r -graph with m edges between V_{i_1}, \dots, V_{i_r} . These are the only edges of G .

Definition 4.10. For $G \in \mathcal{G}(H, n, m, p, \varepsilon)$, let $\#_H(G)$ be the number of “canonical copies” of H in G , meaning that the copy of the vertex i must come from V_i .

Definition 4.11. The r -density $m_r(H)$ of an r -graph H is defined as

$$m_r(H) = \max \left\{ \frac{e(H') - 1}{v(H') - r} : H' \subseteq H \text{ with } v(H') > r \right\}.$$

Now, the hypergraph version of the KLR conjecture is as follows.

Theorem 4.12. For every linear r -graph H (that is, an r -graph where every two edges intersect on at most one vertex) and every $d > 0$, there exist $\varepsilon, \xi > 0$ with the following property. For every $\eta > 0$, there is $C > 0$ such that if $p \geq CN^{-1/m_r(H)}$, then with probability $1 - e^{-\Omega(N^r p)}$ the following holds in $\mathbf{G} \in \mathcal{H}^r(N, p)$. For every $n \geq \eta N$, $m \geq dpn^r$ and every subgraph G' of \mathbf{G} in $\mathcal{G}(H, n, m, p, \varepsilon)$, we have

$$\#_H(G') > \xi \left(\frac{m}{n^r} \right)^{e(H)} n^{v(H)}.$$

The proof of Theorem 4.12 is almost exactly the same as the proof of [8, Theorem 1.6(i)]. In Appendix A we will describe the exact changes one needs to make in order to turn the proof in [8] into a proof of Theorem 4.12.

5 Almost-perfect matchings

For the proof of Lemma 3.6 we will need multiple different ways to find almost-perfect matchings, which we will then be able to complete into perfect matchings using absorbers. The first result we will need is that high-degree subgraphs of upper-quasirandom 3-graphs have almost-perfect matchings.

Lemma 5.1. Let G be an $(p, o(1))$ -upper-quasirandom 3-graph. Let $G' \subseteq G$ be a spanning subgraph with minimum degree at least $0.9p \binom{n}{2}$. Then G' has a matching covering all but $o(n)$ vertices.

Proof. For sufficiently large n' , every n' -vertex 3-graph with minimum degree at least $0.8 \binom{n'}{2}$ has a perfect matching; see for example [14]. So by Lemma 4.4, if we apply the sparse regularity lemma (Lemma 4.2) to G' with small ε and large t_0 , we can find a matching covering $t - 2\sqrt{\varepsilon}t$ vertices of the cluster graph. In each corresponding triple of clusters V_i, V_j, V_q we can greedily find a matching with $(1 - \varepsilon)(n/t)$ vertices, and we can combine these to get a matching in G' covering $n - 3\sqrt{\varepsilon}n$ vertices. Since ε was arbitrary, this implies that we can find a matching covering all but $o(n)$ vertices. \square

The second result we need is that almost-regular 3-graphs can be almost-partitioned into almost-perfect matchings, and moreover the leftover vertices in each matching can be assumed to be “well-distributed”.

Theorem 5.2. Fix $\alpha \in [0, 1]$, and consider a linear 3-graph S with all degrees $\alpha n \pm o(n)$. Then S has $\alpha n - o(n)$ edge-disjoint matchings $M_1, \dots, M_{\alpha n - o(n)}$, each with $n/3 - o(n)$ edges, such that every vertex of S appears in all but $o(n)$ of the M_i .

Theorem 5.2 is a simple consequence of the following theorem of Pippenger and Spencer [37], proved using a Rödl-nibble-type argument, which we reproduce below.

Theorem 5.3. Fix $\alpha \in [0, 1]$, and consider a linear 3-graph S with all degrees $\alpha n \pm o(n)$. Then the edges of S can be partitioned into $\alpha n + o(n)$ edge-disjoint matchings.

Proof of Theorem 5.2. Fix any $\varepsilon > 0$, which we treat as constant. Consider a partition of edge-disjoint matchings as guaranteed by Theorem 5.3. Note that each vertex appears in $\alpha n + o(n)$ of the matchings, by the almost-regularity condition on S . Let Q be the number of matchings with fewer than $n/3 - \varepsilon n$ edges, so that the total number of edges covered by all the matchings is $e(S) \leq Q(n/3 - \varepsilon n) + (\alpha n + o(1) - Q)(n/3) = \alpha n^2/3 - Q\varepsilon n + o(n^2)$. But by the degree condition, we have $e(S) = \alpha n^2/3 + o(n^2)$, so $Q = o(n)$.

Now, deleting the Q matchings with fewer than $n/3 - \varepsilon n$ edges, we obtain a collection of $\alpha n + o(n) - Q = \alpha n - o(n)$ edge-disjoint matchings each with at least $n/3 - \varepsilon n$ edges, such that each vertex appears in $\alpha n + o(n) - Q$ of the matchings, which is all but $o(n)$ of them. Since ε could have been taken arbitrarily small, the desired result follows. \square

6 Packing in good systems

In this section we prove Lemma 3.6. First, we show how to partition our 3-graph into subgraphs with certain “nice” properties.

6.1 Partitioning for packing

The majority of the edges of our 3-graph S will go into subgraphs G_1, \dots, G_ℓ . These subgraphs will have vertex sets U_1, \dots, U_ℓ which each comprise almost all the vertices of S , but they will be rather sparse (each containing approximately a $1/\ell$ fraction of the edges of S). Eventually, we will use Theorem 5.2 to find an almost-perfect packing of almost-perfect matchings in each of these subgraphs.

Some of the remaining edges will go into subgraphs F_1, \dots, F_ℓ , where the vertex set W_i of each F_i is complementary to the vertex set of G_i . Despite each F_i having fewer edges than G_i , the degrees in F_i will be much higher than the degrees in G_i (this will be possible because the W_i will be quite small, and F_i will contain almost all the edges of S within W_i).

Many of the edges that still remain will go into subgraphs H_1, \dots, H_ℓ , whose purpose is to serve as a “bridge” between G_i and F_i . For each $i \leq \ell$, after finding our almost-perfect matchings in G_i , we will use H_i to extend each matching to cover all of G_i and some of F_i , after which we can iteratively complete all of our matchings using absorber-resilience properties of F_i . The details of the properties we will need are summarised in the following lemma. Say that a graph is (R, D) -absorber-resilient if it is D -absorber-resilient after deleting any choice of at most R vertices.

Lemma 6.1. *For any fixed α , consider an $(\alpha, o(1))$ -good linear 3-graph S with n vertices. Fix any (sufficiently small) $\delta > 0$, and let $\ell = \delta^{-5/2}$. Then there exists a constant $\kappa = \kappa(\alpha, \delta) > 0$ and a partition of the edges of S into $3\ell + 1$ subgraphs $G_1, \dots, G_\ell, H_1, \dots, H_\ell, F_1, \dots, F_\ell$, and Q (not necessarily induced or spanning) such that the following properties hold:*

1. **Most of the edges are covered:** at least a $(1 - 3\sqrt{\delta})$ -fraction of the edges of S are in some G_i ;
2. **Controlling the sizes of U_i and W_i :** for each i , the vertex sets $U_i = V(G_i)$ and $W_i = V(F_i)$ partition $V(S)$, and $|W_i| = \delta n + o(n)$;
3. **The 3-graphs G_i are almost regular:** each G_i has all degrees in the range

$$(\alpha(1 - \delta)^2(1 - 2\sqrt{\delta})/\ell)n/2 \pm o(n);$$

4. **The 3-graphs F_i are relatively dense:** each F_i has all degrees at least $0.9999(\alpha\delta^2)n/2$;
5. **Many “bridging” edges:** for every vertex of G_i , there are $\Omega(n)$ edges of H_i with one vertex in U_i and two vertices in W_i ;
6. **The 3-graphs F_i are absorber-resilient:** each F_i is $(\kappa n, 0.9995(\alpha\delta^2)(n/2))$ -absorber-resilient.

Before we prove Lemma 6.1 we briefly state and prove a lemma regarding absorber-resilience.

Lemma 6.2. *For any fixed $\alpha, \delta > 0$, consider an $(\alpha, o(1))$ -good linear 3-graph S with n vertices. There is $\kappa = \kappa(\alpha, \delta) > 0$ such that if U is a random subset of vertices of S , obtained by including each vertex with probability δ independently, then $S[U]$ is $(\kappa n, 0.9995(\alpha\delta^2)(n/2))$ -absorber-resilient with probability at least $1 - e^{-\Omega(n)}$.*

Proof. Choose κ sufficiently small such that $\binom{n}{\kappa n} \exp(-10^{-8}(\delta/2)^4 \alpha^2 n) = e^{-\Omega(n)}$. By the Chernoff bound, we have $|U| \geq \delta n/2$ with probability at least $1 - e^{-\Omega(n)}$. Condition on such an outcome for $|U|$, so that U is now a uniformly random set of vertices of this size. Now, the desired result follows from the robust-absorber resilience property of goodness, taking the union bound over all ways to delete up to κn vertices from U . \square

Now we can prove Lemma 6.1.

Proof of Lemma 6.1. We will describe a random procedure to build the G_i, H_i, F_i and show that the desired properties are satisfied with positive probability. It suffices to show that each of the six properties holds with probability strictly larger than (say) $5/6$ individually; the result will then follow by a simple union bound.

For each $i \leq \ell$ and each $v \in V(S)$, put $v \in W_i$ with probability δ (independently for each i, v). Then, let $U_i = V(S) \setminus W_i$. By a simple application of the Chernoff bounds we see that a.a.s. property 2 holds. Moreover, by Lemma 6.2 we obtain that a.a.s. each $S[W_i]$ is $(\kappa n, 0.9995(\alpha\delta^2)(n/2))$ -absorber-resilient. Since we will choose F_i to be a spanning subgraph of $S[W_i]$, we will obtain that a.a.s. property 6 holds.

Now, we build the G_i, H_i, F_i . Do the following for each edge e , independently.

1. If e is a subset of some W_i (which happens with probability $p_F := 1 - (1 - \delta^3)^\ell \approx \delta^3 \ell = \sqrt{\delta}$), then do the following.
 - (a) If e is a subset of a *unique* W_i , put $e \in F_i$;
 - (b) otherwise, put $e \in Q$.
2. Choose a uniformly random $i \leq \ell$, and choose p_H to satisfy $p_F + p_H = 2\sqrt{\delta}$. If e is not a subset of any W_j , then with probability $p_H/(1 - p_F)$ do the following.
 - (a) If e has one vertex in U_i and two vertices in W_i , put e in H_i .
 - (b) Otherwise, put $e \in Q$.
3. The probability we have not taken any of the previous actions is $p_G := 1 - 2\sqrt{\delta}$. In this case, do the following (still with $i \leq \ell$ as a uniformly random index).
 - (a) If $e \subseteq U_i$, put e in G_i .
 - (b) Otherwise, put e in Q .

By the Chernoff bound, a.a.s. property 1 holds.

Now, for a vertex v , let $d_{G_i}(v)$ be the number of edges $e \in S$ containing v , such that $e \setminus \{v\} \subseteq U_i$ and such that in the above procedure, the random index chosen for e is i . So, if $v \in U_i$ then $d_{G_i} = \deg_{G_i}(v)$. Since S is a partial Steiner triple system, the edges containing v do not intersect other than in v , so $d_{G_i}(v)$ has a binomial distribution $\text{Bin}(\alpha n/2 \pm o(n), (1 - \delta)^2 p_G/\ell)$. By the Chernoff bound and the union bound it follows that a.a.s. property 3 holds.

Next, if for some $i \leq \ell$ and edge $e \in E(S)$ we condition on the event that $e \subseteq W_i$ then the probability that e is contained in some other W_j is $p^* := 1 - (1 - \delta^3)^{\ell-1} \approx \sqrt{\delta}$. So, for every $i \leq \ell$ and vertex v , if we condition on the event that $v \in W_i$ then $\deg_{F_i}(v)$ has a binomial distribution $\text{Bin}(\alpha n/2 \pm o(n), \delta^2(1 - p^*))$, so by the Chernoff bound a.a.s. property 4 holds (provided δ is sufficiently small).

Finally, for every $i \leq \ell$ and vertex v , if we condition on the event that $v \in U_i$ then $\deg_{F_i}(v)$ has a binomial distribution $\text{Bin}(\Omega(n), \Omega(1))$, so by the Chernoff bound a.a.s. property 5 holds. \square

6.2 Absorbers

Absorbers are the basic building blocks for a larger structure which will eventually allow us to complete an almost-perfect matching into a perfect matching. The relative positions of the absorbers in this structure will be determined by a “template” with a “resilient matching” property.

Lemma 6.3. *For any sufficiently large n , there exists a 3-graph T with $10n$ vertices, maximum degree at most 40 , and an identified set Z of $2n$ vertices, such that if we remove any n vertices from Z , the resulting hypergraph has a perfect matching. We call T a resilient template and we call Z its flexible set.*

Lemma 6.3 is an immediate consequence of [31, Lemmas 5.2 and 5.3], and is proved using a construction due to Montgomery [35]. Now, we will want to arrange absorbers in the positions prescribed by a resilient template, as follows.

Definition 6.4. An *absorbing structure* is a 3-graph H of the following form. Consider a resilient template T and put externally vertex-disjoint absorbers on each edge of T , introducing 18 new vertices for each. Then delete the edges of T . That is, the template just describes the relative positions of the absorbers, its edges are not actually in the absorbing structure.

Note that an absorbing structure with a flexible set of size $2n$ has at most $10n + 18 \times 400n/3 = O(n)$ vertices, at most $13 \times (400n/3) = O(n)$ edges and maximum degree at most $40 = O(1)$. An absorbing structure H has the same crucial property as the resilient template T that defines it: if we remove any half of the vertices of the flexible set Z then what remains of H has a perfect matching. Indeed, after this removal we can find a perfect matching M of T , then our perfect matching of H can be comprised of the covering matching of the absorber on each edge of M and the non-covering matching for the absorber on each other edge of T .

So, if we can find an absorbing structure H with flexible set Z in our 3-graph S , then to find a perfect matching it suffices to find a matching that covers all the vertices outside H and any half of the vertices in Z . For the proof of Lemma 3.6, we will be able to construct an absorbing structure with prescribed flexible set using absorber-resilience and the following simple lemma.

Lemma 6.5. *Suppose an n -vertex 3-graph G has the property that in every induced subgraph with $n - n'$ vertices, there is an absorber rooted on every triple of vertices. Suppose also that n' is sufficiently large. Then given any subset $Z \subseteq V(G)$ of size $\omega(1) = |Z| \leq n'/(18 \times 400/3)$, G contains an absorbing structure with flexible set Z .*

Proof. Let $q = |Z|/2$, and fix a resilient template T on an arbitrary set of $10q$ vertices of G (the edges of T do not have to exist in G). Now, we can build our absorbing structure greedily, iteratively finding disjoint absorbers on each edge of T . Indeed, at any point, the non-rooted vertices of the absorbers found so far together comprise a total of at most $18 \times 400q/3 \leq n'$ vertices. After deleting these vertices, we can still continue to find absorbers rooted on every desired triple of vertices. \square

In the proof of Lemma 3.6, to find a perfect matching we will choose Z to be a “rich” set of vertices as guaranteed by Lemma 4.5. Then, we will be able to use Lemma 5.1 to find a matching covering almost all the vertices outside our absorbing structure, and the choice of Z will allow us to extend this to a matching covering all the vertices outside the absorbing structure and half of Z . We will then use the absorbing structure to complete this into a perfect matching.

6.3 Proof of Lemma 3.6

Now, we combine the lemmas in the last two subsections to prove Lemma 3.6.

Proof of Lemma 3.6. Fix sufficiently small $\delta > 0$ (which we will treat as constant for most of the proof). Consider subgraphs $G_1, \dots, G_\ell, H_1, \dots, H_\ell, F_1, \dots, F_\ell$ and vertex sets $U_1, \dots, U_\ell, W_1, \dots, W_\ell$ as guaranteed by Lemma 6.1.

Consider some $i \leq \ell$. Let M_1, \dots, M_s be a collection of $s = \left(\alpha(1 - \delta)^2(1 - 2\sqrt{\delta})/\ell\right)n/2 - o(n)$ almost-perfect matchings in G_i , as guaranteed by Theorem 5.2. We will show that for each $j \leq s$, regardless of how M_1, \dots, M_{j-1} were previously completed, we can complete M_j to a perfect matching using the edges in H_i and F_i that have not been used so far. We will be able to conclude that S has $\alpha(1 - \delta)^2(1 - 2\sqrt{\delta})n/2 - o(n)$ edge-disjoint perfect matchings, which implies the desired result (since δ could have been chosen to be arbitrarily small).

Now, fix an arbitrary ordering of the $o(n)$ vertices in U_i which are not yet covered by M_j . For each such uncovered vertex u in turn, choose an edge of H_i which contains u and two vertices of W_i , which has not already been used for a previous completion and which does not intersect any edges chosen for previous uncovered vertices. Add this edge to M_j . To see that it is possible to make this choice, note that there are $\Omega(n)$ edges in H_i which contain u . Only $o(n)$ of these edges have been used to complete previous matchings, and only $o(n)$ of these edges intersect an edge that was previously chosen for a different uncovered vertex (the edges containing u do not intersect in any vertices other than u , because S is a partial Steiner triple system).

After doing this, M_j covers all of U_i and $o(n)$ vertices of W_i . Let W'_i be the set of unmatched vertices in W_i , and let F'_i be the subgraph consisting of all edges of $F_i[W'_i]$ not used for previous matchings M_q , $q < j$. We now need to find a perfect matching in F'_i .

First note that if δ is sufficiently small, then s (which is about $\alpha\delta^{5/2}n/2$) is much less than the degrees in F_i (which are about $\alpha\delta^2n/2$). So, F_i and F'_i have minimum degree at least $0.9998(\alpha\delta^2)n/2$. Let $\kappa = \Omega(1)$ be as in the statement of Lemma 6.1, let $n' = \min(\kappa n/2, 0.0003(\alpha\delta^2)n/2)$, and note that by the absorber-resilience property of F_i (property 6 in Lemma 6.1), F'_i has the property that in any induced subgraph obtained by deleting at most n' vertices, there is an absorber rooted on every triple of vertices.

By Lemma 4.5 (with sufficiently small ε) and the pseudorandomness property of goodness (which implies the necessary upper-quasirandomness condition), for some $\xi = \Omega(1)$ we can find a vertex set Z' with the following properties:

1. $\Omega(n) \leq |Z'| \leq n'/(2 \times 18 \times 400/3)$;
2. all but $0.01\alpha\delta^2n$ vertices have degree $\Omega(n)$ into Z' ;
3. every subset of $|Z'|(1 - \xi)$ vertices of Z' induces at least one edge.

Arbitrarily add vertices to obtain a set $Z \supseteq Z'$ such that $|Z|/2 = |Z'|(1 - \xi)$. By the absorber-resilience properties of F'_i and Lemma 6.5, we can find an absorbing structure H in F'_i with flexible set Z . Let $X = V(H) \setminus Z$.

Consider the (at most $0.01\alpha\delta^2n$) bad vertices which do not have the guaranteed degree into Z' . Since there are so few of these vertices, and the absorbing structure H is so small compared to the degrees in F'_i , we can greedily find a matching in F'_i avoiding $V(H)$ and covering all the bad vertices. Let F''_i be obtained from F'_i by removing the matched vertices and all the vertices in $X \cup Z'$. Then by Lemma 5.1 we can find a matching covering all but $o(n)$ vertices of F''_i . Let Y be the set of uncovered vertices.

Now, it suffices to find a perfect matching in $F''_i[Y \cup X \cup Z']$. By the richness of Z' , in F''_i we can greedily find a matching of $|Y|$ edges each with a vertex in Y and two vertices in Z' . We can then greedily augment this matching with edges inside Z' until there are $|Z'|(1 - \xi) = |Z|/2$ vertices of Z' uncovered. Finally, we can use the absorbing structure to finish the perfect matching. \square

7 Goodness in random Steiner triple systems

Let \mathcal{S} be a uniformly random ordered $\mathbf{STS}(n)$, and let $N = \binom{n}{2}/3 = (1 + o(1))n^2/6$. We will show that $\mathcal{S}_{\alpha N}$ is a.a.s. $(\alpha, o(1))$ -good as long as α is a sufficiently small constant. This will suffice to prove Lemma 3.5, because we can partition \mathcal{S} into r partial Steiner triple systems which each have the same distribution as $\mathcal{S}_{N/r}$ and then take a union bound.

7.1 Almost-regularity

The almost-regularity property of goodness follows immediately from Theorem 2.4, taking $\mathcal{Q} = \mathcal{O}_{\alpha N}^{\varepsilon, 1}$ and observing that every $S \in \mathcal{O}_{\alpha N}^{\varepsilon, 1}$ has the required property. (We could also give a direct proof by considering a random ordering of the edges of a Steiner triple system; see [31, Section 2.1]).

7.2 Upper-quasirandomness

In this subsection we prove that $\mathcal{S}_{\alpha N}$ a.a.s. satisfies the quasirandomness property of goodness. Consider $q \in \mathbb{N}$ (which we will treat as a sufficiently large constant).

Let $G \sim \mathbb{H}^3(n, p)$ with $p = (\alpha/q)(1 + 10(\alpha/q))/n$. By the Chernoff bound, with probability $1 - e^{-\Omega(n^2)}$ our random 3-graph G has the property that $e_G(X, Y, Z) \leq (\alpha/q)|X||Y||Z|/n + 11(\alpha/q)^2n^2$ for every triple of disjoint vertex subsets X, Y, Z . Denote this property by \mathcal{P} . By Lemma 2.6 (with $S = \emptyset$) and Theorem 2.4 (with no conditioning; that is, $\mathcal{Q} = \mathcal{O}_{\alpha N} \cup \{*\}$), \mathcal{P} also holds a.a.s. in $\mathcal{S}_{\alpha N/q}$. By symmetry, in fact it holds a.a.s. in $\mathcal{S}^{(\ell)} := \mathcal{S}_{\alpha\ell N/q} \setminus \mathcal{S}_{\alpha(\ell-1)N/q}$ for each $\ell \leq q$.

So, a.a.s. for every triple of vertex subsets X, Y, Z in $\mathcal{S}_{\alpha N}$, we have

$$e_{\mathcal{S}_{\alpha N}}(X, Y, Z) = \sum_{\ell \leq q} e_{\mathcal{S}(\ell)}(X, Y, Z) \leq \alpha |X| |Y| |Z| / n + 11(\alpha^2 / q) n^2.$$

Since q could have been arbitrarily large, the desired result follows.

7.3 Embedding Absorbers

To finish the proof of Lemma 3.5, we need to prove that the robust absorber-resilience property of goodness holds a.a.s. in $\mathcal{S}_{\alpha N}$. To do this we use Theorem 2.4 in its full generality, conditioning on the almost-regularity of the first few steps of the triangle removal process.

Let $\alpha' = \alpha/2$, let a be small enough for Theorem 2.4, and let

$$\mathcal{Q} = \{*\} \cup \left\{ S \in \mathcal{O}_{\alpha N} : S_{\alpha' N} \in \mathcal{O}_{\alpha' N}^{n^{-a}, 1} \right\} \supseteq \mathcal{O}_{\alpha N}^{n^{-a}, 1}$$

The plan is to use Lemma 2.6 and Theorem 4.12 to show that for any $S \in \mathcal{O}_{\alpha' N}^{n^{-a}, 1}$, the triangle-removal process $R(S, \alpha' N)$ is extremely likely to produce a partial Steiner triple system which has certain properties which make it easy to find absorbers. We will then be able to use Theorem 2.4 to prove that $\mathcal{S}_{\alpha N}$ is likely to have these properties as well. In combination with some much simpler facts about concentration of degrees in $\mathcal{S}_{\alpha N}$, this will allow us to deduce the desired robust absorber-resilience property. Fix some $\beta > 0$ that is very small compared to α , and for every subset of vertices $W \subseteq [n]$ fix an equipartition $W = \pi_A(W) \cup \pi_B(W)$. Our main objective in this subsection is to prove the following claim.

Claim 7.1. *Consider any $S \in \mathcal{O}_{\alpha' N}^{n^{-a}, 1}$, and let $\mathbf{G} \sim \mathbf{H}^3(n, p)$ with $p = \alpha'(1 + 10\alpha')/n$. If $\beta > 0$ is sufficiently small, then for any subset W of $w \geq \beta n$ vertices, the following holds with probability $1 - e^{-\Omega(n^2)}$. Consider any spanning subgraphs $G' \subseteq \mathbf{G}[W]$ and $S' \subseteq S[W]$ such that*

1. S' has minimum vertex degree at least $0.98(\alpha' w^2 / n^2)(n/2)$, and
2. in G' , for any vertex $v \in W$ and every $\beta^3 n$ -vertex subset U ,

$$\deg_{\pi_A(W) \setminus U}(v), \deg_{\pi_B(W) \setminus U}(v) \geq 0.98 \left(\alpha' \left(\frac{w/2}{n} \right)^2 \right) \frac{n}{2}.$$

Then for any vertices $x, y, z \in W$, there is an absorber in $S' \cup G'$ rooted at x, y, z .

Before proving Claim 7.1 we show how it completes the proof of Lemma 3.5. We break down this deduction into some relatively simple claims. First, in $\mathcal{S}_{\alpha N}$ the degrees into various subsets are typically quite well-behaved. Let $\gamma_w = 0.0001(w/n)^2 \alpha$.

Claim 7.2. *A.a.s. $\mathcal{S}_{\alpha N}$ has the property that for any $w \geq \beta n$ and at least $(1 - \exp(-10^{-8}(w/n)^4 \alpha^2 n)) \binom{n}{w}$ of the w -vertex subsets W :*

1. all the vertex degrees in $\mathcal{S}_{\alpha' N}[W]$ are $(\alpha' w^2 / n^2)(n/2) \pm \gamma_w n$, and
2. in $(\mathcal{S}_{\alpha N} \setminus \mathcal{S}_{\alpha' N})[W]$ every vertex v has degree $(\alpha'(w/2)^2 / n^2)(n/2) \pm \gamma_w n$ into $\pi_A(W)$ and into $\pi_B(W)$, and there are $2 \left(\alpha'(w/2)^2 / n^2 \right) (n/2) \pm \gamma_w n$ edges containing v , a vertex in $\pi_A(W)$ and a vertex in $\pi_B(W)$.

Proof. Let $w \geq \beta n$ and consider a particular w -vertex set W . Observe that randomly reordering the edges and vertices of \mathcal{S} does not change its distribution, so by a concentration inequality for the hypergeometric distribution (see for example [15, Theorem 2.10]) and the union bound, with probability at least $1 - e^{-(3/2)\gamma_w^2 n}$ the desired properties hold for W . By Markov's inequality, a.a.s. for every $w \geq \beta n$ the number of W for which the properties fail to hold is at most $e^{-\gamma_w^2 n} \binom{n}{w} = \exp(-10^{-8}(w/n)^4 \alpha^2 n) \binom{n}{w}$. \square

Next, the following lemma summarises how to use Theorem 2.4 and Lemma 2.6 to turn Claim 7.1 into a fact about random Steiner triple systems.

Claim 7.3. *A.a.s. $\mathcal{S}_{\alpha N}$ has the following property, provided β is sufficiently small. Consider any $w \geq \beta n$ and any w -vertex subset W , and consider spanning subgraphs $S'' \subseteq (\mathcal{S}_{\alpha N} \setminus \mathcal{S}_{\alpha' N})[W]$ and $S' \subseteq \mathcal{S}_{\alpha' N}[W]$ such that*

1. S' has minimum vertex degree at least $0.98(\alpha' w^2/n^2)(n/2)$, and
2. in S'' , for any vertex $v \in W$ and every $\beta^3 n$ -vertex subset U , we have

$$\deg_{\pi_A(W) \setminus U}(v), \deg_{\pi_B(W) \setminus U}(v) \geq 0.98 \left(\alpha' \left(\frac{w/2}{n} \right)^2 \right) \frac{n}{2}.$$

Then for any vertices $x, y, z \in W$, there is an absorber in $S' \cup S''$ rooted at x, y, z .

Proof. Note that the property in Claim 7.1 is a monotone decreasing property of \mathbf{G} (depending on S). Then, combine Lemma 2.6, Theorem 2.4, and Claim 7.1 (with \mathcal{Q} as in the beginning of this subsection). \square

Now, given Claim 7.2 and Claim 7.3 it is fairly immediate to deduce that $\mathcal{S}_{\alpha N}$ a.a.s. has the desired absorber-resilience property, as follows. Suppose that $\mathcal{S}_{\alpha N}$ satisfies the conclusions of Claim 7.2 and Claim 7.3, and consider $w \geq \beta n$. Suppose for some w -vertex subset W that $\mathcal{S}_{\alpha' N}[W]$ and $(\mathcal{S}_{\alpha N} \setminus \mathcal{S}_{\alpha' N})[W]$ both satisfy the degree conditions in Claim 7.2 (this is true for at least $(1 - \exp(-10^{-8}(w/n)^4 \alpha^2 n)) \binom{n}{w}$ of the choices of W). We want to show that $\mathcal{S}_{\alpha N}[W]$ is D -absorber-resilient, for $D = 0.999(\alpha w^2/n^2)(n/2)$. Consider a subgraph $S \subseteq \mathcal{S}_{\alpha N}[W]$ with minimum degree D . Observe that by the choice of γ_w , the hypergraph $S' = S \cap \mathcal{S}_{\alpha' N}$ consisting of the edges of S that appear in $\mathcal{S}_{\alpha' N}$ has minimum degree at least

$$0.999 \left(\alpha \frac{w^2}{n^2} \right) \frac{n}{2} - 4 \left(\left(\alpha' \left(\frac{w/2}{n} \right)^2 \right) \frac{n}{2} \pm \gamma_w n \right) \geq 0.98 \left(\alpha' \frac{w^2}{n^2} \right) \frac{n}{2}.$$

Next, let $S'' = S \setminus \mathcal{S}_{\alpha' N}$, consider any vertex v and any set of U of $\beta^3 n$ other vertices of W . Since S is a partial Steiner triple system there are at most $\beta^3 n$ edges of S involving both v and U , so for sufficiently small β , in S'' we have

$$\begin{aligned} \deg_{\pi_A(W) \setminus U}(v) &\geq 0.999 \left(\alpha' \frac{w^2}{n^2} \right) \frac{n}{2} - \left(\left(\alpha' \frac{w^2}{n^2} \right) (n/2) \pm \gamma_w n \right) - 3 \left(\left(\alpha' \left(\frac{w/2}{n} \right)^2 \right) \frac{n}{2} \pm \gamma_w n \right) - \beta^3 n \\ &\geq 0.98 \left(\alpha' \left(\frac{w/2}{n} \right)^2 \right) \frac{n}{2}, \end{aligned}$$

and similarly $\deg_{\pi_B(W) \setminus U}(v) \geq 0.98 \left(\alpha' (w/2)^2/n^2 \right) (n/2)$. So, in S' there is an absorber on every triple of vertices, concluding the proof of D -absorber-resilience.

It remains to prove Claim 7.1. As in the statement of Claim 7.1, consider any $S \in \mathcal{O}_{\alpha' N}^{n-a, 1}$ and let $\mathbf{G} \sim H^3(n, p)$ with $p = \alpha'(1 + 10\alpha')/n$. The proof of Claim 7.1 will reduce to another claim about a family of auxiliary graphs obtained by contracting edges of subgraphs of \mathbf{G} . For a vertex set W containing a vertex v , let $H_{S[W]}(v)$ be the set of all edges containing v in $S[W]$.

Definition 7.4. Consider any subset W of $w \geq \beta n$ vertices of \mathbf{G} , fix any $x, y, z \in W$, and consider any subsets $H_x \subseteq H_{S[W]}(x)$, $H_y \subseteq H_{S[W]}(y)$, $H_z \subseteq H_{S[W]}(z)$ each with $0.98(\alpha' w^2/n^2)(n/2)$ edges. Let $\mathbf{G}(W, x, y, z, H_x, H_y, H_z)$ be a 3-graph obtained from $\mathbf{G}[W \setminus \{x, y, z\}]$ by doing the following.

1. First, for $k = \beta^4 n$, choose distinct vertices

$$a_1^x, \dots, a_k^x, b_1^x, \dots, b_k^x, a_1^y, \dots, a_k^y, b_1^y, \dots, b_k^y, a_1^z, \dots, a_k^z, b_1^z, \dots, b_k^z$$

such that for each $0 \leq i \leq k$, we have $\{x, a_i^x, b_i^x\} \in H_x$, $\{y, a_i^y, b_i^y\} \in H_y$, $\{z, a_i^z, b_i^z\} \in H_z$. If β is sufficiently small this can be done greedily, recalling that the edges of H_x (respectively, of H_y or of H_z) intersect only at x (respectively, only at y or at z).

2. Let W_A (respectively W_B) be the subset of $\pi_A(W)$ (respectively $\pi_B(W)$) obtained by removing x, y, z and all $3k$ of the vertices chosen in the first step.
3. Then, for each $v \in \{x, y, z\}$, delete all edges containing a vertex of H_v , except those containing two vertices from W_A and some a_i^v , or two vertices from W_B and some b_i^v .
4. Finally, for each $v \in \{x, y, z\}$, contract each pair $\{a_i^v, b_i^v\}$ to a single vertex. Let U_v be the set of newly contracted vertices.

Now, to prove Claim 7.1 it suffices to prove the following claim (observe that $2|U_x| + 2|U_y| + 2|U_z| + 3 \leq \beta^3 n$ for sufficiently small β , and taking the union bound over all choices of $W, x, y, z, H_x, H_y, H_z$ costs us a factor of only $e^{O(n)}$).

Claim 7.5. *For any $W, x, y, z, H_x, H_y, H_z$ as in Definition 7.4, $\mathbf{G}' := \mathbf{G}(W, x, y, z, H_x, H_y, H_z)$ has the following property with probability $1 - e^{-\Omega(n^2)}$. For any spanning subgraphs*

$$G_A \subseteq \mathbf{G}'[W_A \cup U_x \cup U_y \cup U_z], \quad G_B \subseteq \mathbf{G}'[W_B \cup U_x \cup U_y \cup U_z]$$

with minimum vertex degree at least $0.98 \left(\alpha(w/2)^2 / n^2 \right) (n/2)$, there are vertices $x' \in U_x, y' \in U_y, z' \in U_z$ such that in both G_A and G_B there is a sub-absorber rooted on x', y', z' .

We will prove Claim 7.5 with the sparse regularity lemma (Lemma 4.2) and Theorem 4.12. Before doing this, we prove some final auxiliary lemmas. First, we need to know that absorbers are suitably sparse to apply Theorem 4.12. Recall the definition of 3-density from Definition 4.11.

Lemma 7.6. *Consider an absorber and contract each rooted edge to a single vertex. Call the resulting 3-graph H a contracted absorber. (H is equivalently obtained by gluing together two sub-absorbers on their rooted vertices). Then H has $m_3(H) < 1$.*

Proof. Note that H has maximum degree 2. Let H' be a subgraph of H with q vertices of degree 2. Then, $3e(H') \leq v(H') + q$, and it follows that we can only have $(e(H') - 1)/(v(H') - 3) \geq 1$ if $q \geq 2v(H') - 6$. This is impossible if $v(H') > 6$, so it suffices to consider subgraphs H' with up to 6 vertices. It then suffices to check by hand that every pair of edges spans at least 5 vertices, and every triple of edges spans at least 7 vertices. \square

Second, the sparse regularity lemma will give us a very dense cluster 3-graph, and we will need to be able to find absorbers in such a 3-graph.

Lemma 7.7. *Let F be an n -vertex 3-graph with degrees at least $0.96 \binom{n}{2}$, having an identified subset U of at most $0.001n$ vertices. If n is sufficiently large, then for any $x, y, z \in U$ there is a sub-absorber rooted at x, y, z in which the only edges involving vertices of U are the rooted ones.*

Proof. Let $V = V(F)$ and consider any $x, y, z \in U$. Every vertex has degree at least $0.96 \binom{n}{2} - 0.001n^2$ into $V \setminus U$, so there are at least $(0.96 \binom{n}{2} - 0.001n^2)^3 - o(n^6)$ choices of three disjoint edges each containing one of x, y and z and two vertices in $V \setminus U$. For each such choice of three edges, there is a pair of edges in U whose presence would yield a suitable sub-absorber, and a pair of edges in $F[V \setminus U]$ can contribute in this way to at most $(3!)^2$ sub-absorbers. It follows that there are at least $(0.96 \binom{n}{2} - 0.001n^2)^3 / (3!)^2 - o(n^6)$ pairs of edges whose presence would yield a sub-absorber. But note that all but at most $(1 - 0.96^2) \binom{n}{3}^2$ of the possible pairs of edges are present in F , and $(0.96/2! - 0.001)^3 / (3!)^2 > (1 - 0.96^2)(1/3!)^2$, so for large n , there must be a sub-absorber as desired. \square

Now we can finally prove Claim 7.5, completing the proof of Lemma 3.5.

Proof of Claim 7.5. Consider all the triples of vertices intersecting $U_x \cup U_y \cup U_z$, except those with one vertex in $U_x \cup U_y \cup U_z$ and two vertices in W_A , or one vertex in $U_x \cup U_y \cup U_z$ and two vertices in W_B . None of these triples are edges in \mathbf{G}' . Let $\mathbf{G}'_{\text{extra}}$ be a random 3-graph, independent from \mathbf{G}' , where each of these triples is included with probability $p := \alpha'(1 + 10\alpha')/n$ independently. Now $\mathbf{G}' \cup \mathbf{G}'_{\text{extra}}$ is a standard binomial random 3-graph on the vertex set $U_x \cup U_y \cup U_z \cup W_A \cup W_B$ where each edge is present

with probability p . The only purpose of the edges in $\mathbf{G}_{\text{extra}}$ is to put us in the setting for Theorem 4.12; we will not actually use these edges for anything.

Let H be a contracted absorber, as defined in Lemma 7.6. By Theorem 4.12 there is $\varepsilon > 0$ such that with probability $1 - e^{-\Omega(n^2)}$ the random 3-graph $\mathbf{G}' \cup \mathbf{G}_{\text{extra}}$ has the property that for any $n' = \Omega(n)$ and any $m \geq 0.001p(n')^3$, every subgraph $G'' \subseteq \mathbf{G}' \cup \mathbf{G}_{\text{extra}}$ in $\mathcal{G}(H, n', m, p, \varepsilon)$ has $\#_H(G'') > 0$. Also, by the Chernoff bound (basically as in Section 7.2), \mathbf{G} is (p, β) -upper-quasirandom with probability $1 - e^{-\Omega(n^2)}$.

Assuming that the above two properties hold, apply Lemmas 4.2 and 4.6 to $G_A \cup G_B$ with regularity parameter $\varepsilon > 0$ which is small even compared to β , and with $t_0 \geq 1/\varepsilon$, to obtain a partitioned t -vertex cluster 3-graph \mathcal{R} with threshold 0.001. Let $\mathcal{U}_x, \mathcal{U}_y, \mathcal{U}_z, \mathcal{W}_A, \mathcal{W}_B$ be the sets of clusters fully contained in U_x, U_y, U_z, W_A, W_B . By Lemma 4.8, in both $\mathcal{R}_A := \mathcal{R}[\mathcal{W}_A \cup \mathcal{U}_x \cup \mathcal{U}_y \cup \mathcal{U}_z]$ and in $\mathcal{R}_B := \mathcal{R}[\mathcal{W}_B \cup \mathcal{U}_x \cup \mathcal{U}_y \cup \mathcal{U}_z]$, all but $\sqrt{\varepsilon}t$ vertices have degree at least $0.97 \binom{t/2}{2}$. Delete at most $2\sqrt{\varepsilon}t$ vertices to obtain induced subgraphs $\mathcal{R}'_A, \mathcal{R}'_B$ with minimum degree at least $0.96 \binom{t/2}{2}$.

Now, fix clusters $V_x \in \mathcal{U}_x, V_y \subseteq \mathcal{U}_y, V_z \subseteq \mathcal{U}_z$ which appear in both $\mathcal{R}'_A, \mathcal{R}'_B$, and apply Lemma 7.7 to \mathcal{R}'_A and to \mathcal{R}'_B to find sub-absorbers in the cluster graph rooted at V_x, V_y, V_z . These two sub-absorbers comprise a contracted absorber \mathcal{A} in \mathcal{R} , so the edges between the clusters in $V(\mathcal{A})$ give us a subgraph of \mathbf{G}' in $\mathcal{G}(H, n', m, p, \varepsilon)$, for $n' = n/t \geq n/T$ and $m \geq 0.001p(n')^3$. It follows that we can find a canonical copy of a contracted absorber H , giving us a sub-absorber in G_A and in G_B rooted on the same three vertices. \square

8 Concluding remarks

There are many interesting further directions of research regarding random Steiner triple systems. Most obviously, Conjecture 1.1 is still open, though we imagine that an exact result would be quite difficult to prove. Perhaps a good starting point would be the methods of Kühn and Osthus [30], and Knox, Kühn and Osthus [25] for perfectly packing Hamilton cycles in random graphs and random tournaments.

A second interesting direction is to study the discrepancy of random Steiner triple systems. Combining the ideas in Section 7.2 and [31, Section 5.1.2], we have essentially proved the following theorem bounding the discrepancy of a random Steiner triple system, which may be of independent interest.

Theorem 8.1. *Let \mathcal{S} be a uniformly random order- n Steiner triple system. Then \mathcal{S} a.a.s. satisfies the following property. For every triple of vertex subsets X, Y, Z , we have $e(X, Y, Z) = |X||Y||Z|/n + o(n^2)$.*

We remark that an analogous theorem for Latin squares (with a stronger error term) was proved by Kwan and Sudakov [32]. See also the related conjectures in [33]. It would be very interesting if one could substantially improve the error term $o(n^2)$; we imagine the correct order of magnitude is $O\left(\sqrt{|X||Y||Z|}\right)$, but a proof of this would require substantial new ideas.

Next, another interesting direction concerns containment and enumeration of subgraphs. Using Theorem 4.12 and the sparse regularity lemma (Lemma 4.2) in combination with Theorem 2.4 and Lemma 2.6, it is straightforward to prove the following result.

Theorem 8.2. *Let H be a 3-graph with $m_3(H) < 1$. Then there is $\xi = \xi(H) > 0$ such that a uniformly random $\text{STS}(n)$ a.a.s. contains at least $\xi n^{v(H)-e(H)}$ copies of H .*

We imagine that actually the number of (labelled) copies of H should a.a.s. be $(1 \pm o(1))n^{v(H)-e(H)}$, but it is less obvious how to prove this. In particular, due to the ‘‘infamous upper tail’’ issue (see [16]) and the fact that Theorem 2.4 only works with properties that hold with probability extremely close to 1, any kind of upper bound on subgraph counts would require new ideas. We also remark that a lower bound was proved by Simkin [42] in the special case where H is a *Pasch configuration*, using ideas from [31].

Another interesting question (also mentioned in [31]) is whether a random Steiner triple system typically contains a Steiner triple subsystem on fewer vertices. McKay and Wanless [34] proved that almost all Latin squares have many small Latin subsquares (see also [32]), but it was conjectured by Quackenbush [38] that most Steiner triple systems do not have proper subsystems. By comparison with a binomial random 3-graph, it seems likely that this conjecture is actually false, but it seems that substantial new ideas would be required to prove or disprove it.

Finally, one might try to generalise from Steiner triple systems to other classes of designs. It seems that the arguments in this paper should generalise in a straightforward fashion to Latin squares, proving that a random order- n Latin squares a.a.s. has $n - o(n)$ disjoint transversals (see [31] for a definition of Latin squares, a discussion of how the methods in Section 2 generalise to random Latin squares, and a discussion of the significance of transversals in Latin squares). It was actually conjectured by van Rees [43] that a random order- n Latin square typically does *not* have a decomposition into n disjoint transversals, though Wanless and Webb [44] observed that numerical observations seem more in line with the Latin squares analogue of Conjecture 1.1.

Also, a (q, r, λ) -*design* ($q > r$) of order n is a q -uniform hypergraph on the vertex set $[n]$ such that every r -set of vertices is included in exactly λ edges. A (q, r) -*Steiner system* is a $(q, r, 1)$ -design (so, a Steiner triple system is a $(3, 2, 1)$ -design or equivalently a $(3, 2)$ -Steiner system, and a d -regular graph is a $(2, 1, d)$ -design). We expect that it should be fairly routine to adapt the definition of an absorber in the obvious way to prove that almost all (q, r, λ) -designs have a decomposition of almost all their edges into perfect matchings. As for Steiner triple systems, a design is said to be resolvable if it admits a decomposition into perfect matchings, and the general problem of whether resolvable block designs exist was only very recently solved by Keevash [21].

However, note that a 3-uniform perfect matching is actually a $(3, 1)$ -Steiner system, so as a sweeping generalisation of Conjecture 1.1 we might ask for which $r' \leq r$ and $\lambda' \leq \lambda$ do (q, r, λ) -designs typically admit a decomposition into spanning (q, r', λ') -designs of the same order. We note that in the case of regular graphs a much stronger phenomenon occurs: there is a sense in which a random $(d_1 + d_2)$ -regular graph is “asymptotically the same” as a random d_1 -regular graph combined with a random d_2 -regular graph (provided $d_1 + d_2 > 2$; see [15, Section 9.5]).

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A Adapting the proof of the KLR Conjecture to linear hypergraphs

In their paper [8] on the KLR conjecture for graphs, Conlon, Gowers, Samotij and Schacht explicitly mentioned that their methods should extend to hypergraphs, and that the generalisation should be particularly simple in the case of linear hypergraphs. Actually, the generalisation is so simple that we can describe in this short appendix exactly what changes to make to their proof of their Theorem 1.6 (i) (appearing in Section 2 of their paper), to turn it into a proof of Theorem 4.12. All theorem/lemma/section references are with respect to [8], unless noted otherwise.

- Change the notation “ $G(H)$ ” to “ $\#_H(G)$ ” (we changed this notation to avoid confusion with other notation in the paper).
- Change every instance of “graph” to “ r -graph” and every instance of “bipartite graph” to “ r -partite r -graph”.
- Change every instance of “ m_2 ” to “ m_r ”.
- Change every instance of “ n^2 ” to “ n^r ”, every instance of “ $n^{v(H)-2}$ ” to “ $n^{v(H)-r}$ ”, and every instance of “ N^2 ” to “ N^r ”.
- The definition of (ε, d) -lower-regularity preceding the statement of Theorem 2.1 should be changed in the obvious way: an r -partite r -graph between sets V_1, \dots, V_r is (ε, d) -lower-regular if, for every $V'_1 \subseteq V_1, \dots, V'_r \subseteq V_r$ with $|V'_i| \geq \varepsilon|V_i|$, the density $d(V'_1, \dots, V'_r)$ of edges between V'_1, \dots, V'_r satisfies $d(V'_1, \dots, V'_r) \geq d$.
- In the statement of Theorem 2.1, change “let H be an arbitrary graph” to “let H be an arbitrary linear r -graph”.
- At the beginning of the deduction of Theorem 1.6 (i) from Theorem 2.1, change the observation “ $m_2(H) \geq 1$ ” to “ $m_r(H) \geq 1/(r-1)$ ”.
- In the second displayed equation in the proof of Theorem 1.6 (i), change “ $e_{G_{N,p}}(W_i, W_j) \geq 2pn^2$ for some $ij \in E(H)$ ” to “ $e_{G_{N,p}}(W_{i_1}, \dots, W_{i_r}) \geq 2pn^r$ for some $\{i_1, \dots, i_r\} \in E(H)$ ”.
- A one-sided counting lemma is presented without proof as Lemma 2.4. The corresponding generalisation for linear hypergraphs (with “every graph H ” replaced with “every linear r -graph H ”) follows from essentially the same proof as [28, Lemma 10] (see also [9, Lemma 22]).

- In Section 2.2, change the definition of C from “ $32RL^RC'/(\varepsilon^2d)$ ” to “ $16rRL^RC'/(\varepsilon^2d)$ ”, and in the proof of Claim 2.5 change the last display from “ $e(H) \cdot 2^{2n} \exp(-\varepsilon^2 dp_s n^2/16)$ ” to “ $e(H) \cdot 2^{rn} \exp(-\varepsilon^2 dp_s n^r/16)$ ”.
- In Section 2.2.2, the definition of Z_s should be generalised in the obvious way:

$$Z_s = \left\{ e \in G(V_{i_1}, \dots, V_{i_r}) : \deg_{H''}(e, G, G'_s) \geq \frac{\xi'}{2} p_s^{\varepsilon(H'')} n^{v(H)-r} \right\},$$

where $\{i_1, \dots, i_r\}$ is the edge of H' that is missing in H'' , and V_{i_1}, \dots, V_{i_r} are the subsets of $V(G)$ corresponding to the vertices i_1, \dots, i_r .

- In Case 1 of the proof of Claim 2.6, change “there exist sets $X_i \subseteq V_i$ and $X_j \subseteq V_j$ with $|X_i|, |X_j| \geq \varepsilon n$ ” to “there exist sets $X_{i_1} \subseteq V_{i_1}, \dots, X_{i_r} \subseteq V_{i_r}$ with $|X_{i_1}|, \dots, |X_{i_r}| \geq \varepsilon n$ ”. Also, change all instances of “ (X_i, X_j) ” to “ $(X_{i_1}, \dots, X_{i_r})$ ” and change all instances of “ $|X_i||X_j|$ ” to “ $|X_{i_1}| \dots |X_{i_r}|$ ”.
- In Case 2 of the proof of Claim 2.6, change “for every $i', j' \in E(H)$ and every pair of sets $W_{i'} \subseteq V_{i'}$ and $W_{j'} \subseteq V_{j'} \in$ with $|W_{i'}| \geq \varepsilon n$ and $|W_{j'}| \geq \varepsilon n$ ” to “for every $i'_1, \dots, i'_r \in E(H)$ and any choice of $W_{i'_1} \subseteq V_{i'_1}, \dots, W_{i'_r} \subseteq V_{i'_r}$ with $|W_{i'_1}|, \dots, |W_{i'_r}| \geq \varepsilon n$ ”. Also, change all instances of “ $(W_{i'}, W_{j'})$ ” to “ $(W_{i'_1}, \dots, W_{i'_r})$ ”, all instances of “ $|W_{i'}||W_{j'}|$ ” to “ $|W_{i'_1}| \dots |W_{i'_r}|$ ”, and all instances of “ (V_i, V_j) ” to “ $(V_{i_1}, \dots, V_{i_r})$ ”.