

# Ramsey graphs induce subgraphs of quadratically many sizes

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## Abstract

An  $n$ -vertex graph is called  $C$ -Ramsey if it has no clique or independent set of size  $C \log n$ . All known constructions of Ramsey graphs involve randomness in an essential way, and there is an ongoing line of research towards showing that in fact all Ramsey graphs must obey certain “richness” properties characteristic of random graphs. Motivated by an old problem of Erdős and McKay, recently Narayanan, Sahasrabudhe and Tomon conjectured that for any fixed  $C$ , every  $n$ -vertex  $C$ -Ramsey graph induces subgraphs of  $\Theta(n^2)$  different sizes. In this paper we prove this conjecture.

## 1 Introduction

An induced subgraph of a graph is said to be *homogeneous* if it is a clique or independent set. A classical result in Ramsey theory, proved in 1935 by Erdős and Szekeres [16], is that every  $n$ -vertex graph has a homogeneous subgraph with at least  $\frac{1}{2} \log_2 n$  vertices. On the other hand, Erdős [14] famously used the probabilistic method to prove that there exists an  $n$ -vertex graph with no homogeneous subgraph on  $2 \log_2 n$  vertices. Despite significant effort (see for example [20, 7, 12, 11]), there are no non-probabilistic constructions of graphs with comparably small homogeneous sets.

For some fixed  $C$ , say a graph is  $C$ -Ramsey if it has no homogeneous subgraph of size  $C \log_2 n$ . It is widely believed that  $C$ -Ramsey graphs must in some sense resemble random graphs, and this belief has been supported by a number of theorems showing that certain “richness” properties characteristic of random graphs hold for all  $C$ -Ramsey graphs. The first result of this type was due to Erdős and Szemerédi [17], who showed that  $C$ -Ramsey graphs have density bounded away from 0 and 1. Further research has focused on showing that certain statistics or substructures can take many different values. Improving a result of Erdős and Hajnal [15], Prömel and Rödl [23] proved that for every constant  $C$  there is  $c > 0$  such that every  $n$ -vertex  $C$ -Ramsey graph contains every possible graph on  $c \log_2 n$  vertices as an induced subgraph. Shelah [24] proved that every  $n$ -vertex  $C$ -Ramsey graph contains  $2^{\Omega(n)}$  non-isomorphic induced subgraphs. Fairly recently, answering a question of Erdős, Faudree and Sós [18, 19], Bukh and Sudakov [9] showed that every  $n$ -vertex  $C$ -Ramsey graph has an induced subgraph with  $\Omega(\sqrt{n})$  different degrees.

Two significant open problems in this area concern variation in the numbers of edges and vertices in induced subgraphs. For a graph  $G$ , let

$$\begin{aligned}\Phi(G) &= \{e(H) : H \text{ is an induced subgraph of } G\}, \\ \Psi(G) &= \{(v(H), e(H)) : H \text{ is an induced subgraph of } G\}.\end{aligned}$$

Erdős and McKay [18, 19] conjectured that for any  $C$  there is  $\delta > 0$  such that for every  $n$ -vertex  $C$ -Ramsey graph  $G$ , the set  $\Phi(G)$  contains the interval  $\{0, \dots, \delta n^2\}$ . Erdős, Faudree and Sós [18, 19]

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conjectured that for any fixed  $C$  and any  $n$ -vertex  $C$ -Ramsey graph  $G$ , we have  $|\Psi(G)| = \Omega(n^{5/2})$ . The best progress on the former conjecture is due to Alon, Krivelevich and Sudakov [4], who proved it with  $n^\delta$  in place of  $\delta n^2$ . The best progress on the latter conjecture is due to Alon, Balogh, Kostochka and Samotij [1] (improving work of Alon and Kostochka [3]), who proved it with 2.369 in place of 5/2. We also remark that strengthenings of both these conjectures have been shown to hold for random graphs [10, 3].

Recently, Narayanan, Sahasrabudhe and Tomon [21] proposed a natural weakening of the aforementioned Erdős–McKay conjecture in the spirit of the Erdős–Faudree–Sós conjecture. Specifically, they conjectured that  $|\Phi(G)| = \Omega(n^2)$  for every  $n$ -vertex  $C$ -Ramsey graph  $G$ , and proved the weaker result that  $|\Phi(G)| = n^{2-o(1)}$  (to be precise, they explain that their methods actually give a bound of the form  $n^2/e^{\Theta(\sqrt{\log n})}$ ). In this paper we prove Narayanan, Sahasrabudhe and Tomon’s conjecture.

**Theorem 1.** *For any fixed  $C$ , and any  $n$ -vertex  $C$ -Ramsey graph  $G$ , we have  $|\Phi(G)| = \Omega(n^2)$ .*

We remark that the order of magnitude  $n^2$  is best possible, because  $\Phi(G) \subseteq \{0, \dots, \binom{n}{2}\}$  for any  $n$ -vertex graph. Very loosely speaking, the general approach of our proof is similar to the proof in [21], but we make a number of simplifications and introduce some new ideas that we hope will be useful for other problems.

## 1.1 Notation and basic definitions

We use standard asymptotic notation throughout, and all asymptotics are as  $n \rightarrow \infty$  unless stated otherwise. Floor and ceiling symbols will be systematically omitted where they are not crucial.

For two multisets  $A$  and  $B$ , let  $A \Delta B$  be the set of elements which have different multiplicities in  $A$  and  $B$  (so if  $A$  and  $B$  are ordinary sets, then  $A \Delta B$  is the ordinary symmetric difference  $(A \setminus B) \cup (B \setminus A)$ ). For a set  $A$ , we denote by  $\binom{A}{2}$  the set of all unordered pairs of elements of  $A$ .

We also use standard graph theoretic notation throughout. In particular, in a graph, the density of a set of vertices  $A$  is defined as  $d(A) = e(A)/\binom{|A|}{2}$ , where  $e(A)$  is the number of edges which are contained inside  $A$ . Similarly, for any two sets  $A$  and  $B$ , the density between them is  $d(A, B) = e(A, B)/|A||B|$ , where  $e(A, B)$  is the number of edges between  $A$  and  $B$ . For a vertex  $v$  and a set of vertices  $A$ , we denote the set of neighbours of  $v$  in  $A$  by  $N_A(v) = N(v) \cap A$  and we denote the degree of  $v$  into  $A$  by  $d_A(v) = |N_A(v)|$ .

We also make some less standard graph theoretic definitions that will be convenient for the proof. For a pair of vertices  $\mathbf{v} = \{v_1, v_2\}$ , let  $N(\mathbf{v})$  (respectively  $N_U(\mathbf{v})$ ) be the multiset union of  $N(v_1)$  and  $N(v_2)$  (respectively, of  $N_U(v_1)$  and  $N_U(v_2)$ ). Let  $d(\mathbf{v}) = d(v_1) + d(v_2)$  (respectively  $d_U(\mathbf{v}) = d_U(v_1) + d_U(v_2)$ ) be the size of  $N(\mathbf{v})$  (respectively, of  $N_U(\mathbf{v})$ ), accounting for multiplicity.

## 2 Ideas of the proof and previous work

As mentioned in the introduction, our proof builds on some ideas of Narayanan, Sahasrabudhe and Tomon in [21]. This work in turn builds on the ideas of Bukh and Sudakov in [9]. In this section we briefly outline the relevant ideas in both these papers, and discuss the new ideas in this paper.

In [9], Bukh and Sudakov proved that  $n$ -vertex Ramsey graphs have subgraphs with  $\Omega(\sqrt{n})$  distinct degrees. To do this, they introduced the notion of *diversity*, as follows. Say an  $n$ -vertex graph is  $(c, \gamma, )$ -diverse if for each vertex  $x \in V$ , we have  $|N(x) \Delta N(y)| < cn$  for at most  $n^\gamma$  vertices  $y \in V$ . Roughly speaking, this means the neighbourhoods of most pairs of vertices are very different. Bukh and Sudakov went on to prove that for any  $C$  and  $\gamma > 0$ , all  $C$ -Ramsey graphs have  $(\Omega(1), \gamma)$ -diverse induced subgraphs of linear size.

Now, in an  $n$ -vertex  $(c, \gamma)$ -diverse graph  $G$ , consider a random vertex subset  $U$  obtained by including each vertex with some fixed probability  $p$  independently. By the diversity assumption, for most pairs of vertices  $u, v$  their degrees  $d_U(u), d_U(v)$  into  $U$  are not too strongly correlated, and the probability they are exactly equal turns out to be  $O(1/\sqrt{n})$ . (A simple intuitive reason for this probability is that  $d_U(u) - d_U(v)$  is approximately normally distributed with standard deviation  $\Theta(\sqrt{n})$ ). One may then compute that the expected number of pairs of vertices with the same degree into  $U$  is  $O(n^{3/2} + n^{1+\gamma})$ , so provided  $\gamma < 1/2$ , one may use Turán's theorem to show that there is an outcome of  $G[U]$  with  $\Omega(\sqrt{n})$  different degrees. This fact has some immediate consequences for  $|\Phi(G)|$ : for example, Bukh and Sudakov observed that one can obtain  $\Omega(\sqrt{n})$  subgraphs with different numbers of edges simply by choosing different vertices of  $U$  to delete from  $G[U]$ .

There are two straightforward ways one might hope to improve on this simple bound. First, we can repeat the above argument for many different values of  $p$ , and second, instead of deleting single vertices, we might hope to obtain a richer variety of subgraphs by adding and deleting different combinations of vertices. Narayanan, Sahasrabudhe and Tomon [21] combined both these ideas, as follows.

In an  $n$ -vertex  $(c, \gamma)$ -diverse graph  $G$ , first use the pigeonhole principle to identify a set  $W_0$  of  $\Theta(\sqrt{n})$  vertices with degrees contained in a narrow interval  $[d, d + \sqrt{n}]$ , for some  $d = \Omega(n)$ . Then, for  $\Theta(\sqrt{n})$  well-separated values of  $p$ , do the following. Let  $U$  be a random subset of the vertices not in  $W_0$ , obtained by including each vertex with probability  $p$ . Using the diversity of  $G$ , one may compute that the expected number of pairs of vertices of  $W_0$  which have the same degree into  $U$  is  $O(|W_0|^2/\sqrt{n} + |W_0|n^\gamma) = O(n^{1/2+\gamma})$ , so one can show with Turán's theorem that there is an outcome of  $U$  such that  $W_0$  contains a subset  $W$  of  $\Omega(n^{1/2-\gamma})$  vertices with different degrees into  $U$ . Moreover, since the initial degrees  $d(w)$  were chosen to be very similar, one can show that actually the  $d_U(w)$  are likely to still lie in an interval of length  $O(\sqrt{n})$ .

Because the degrees of vertices in  $W$  are so well-behaved, one can then show that many different values  $e(G[U \cup Z])$  can be obtained with different subsets  $Z \subseteq W$ . Indeed, by varying the number of vertices in  $Z$ , one can change  $e(G[U \cup Z])$  by increments of  $\Theta(n)$ , and by swapping low-degree vertices with high-degree vertices, one can change  $e(G[U \cup Z])$  by increments of about  $\sqrt{n}$ . That is to say, by choosing subsets  $Z \subseteq W$  of certain types, one can obtain  $\Omega(n^{1-2\gamma})$  different values of  $e(G[U \cup Z])$  separated by  $\Omega(\sqrt{n})$  from each other.

The above ideas yield  $|\Phi(G)| = \Omega(n^{3/2-2\gamma})$  in a relatively straightforward fashion. Since the diversity lemma of Bukh and Sudakov allows  $\gamma$  to be arbitrarily small, this proves that  $n$ -vertex  $O(1)$ -Ramsey graphs  $G$  have  $|\Phi(G)| = n^{3/2-o(1)}$ . In order to improve this to  $n^{2-o(1)}$ , one would ideally like to be able to show that for each of the  $\Omega(n^{1-o(1)})$  choices of  $Z$  described above, one can add an additional vertex  $w \in W$  to  $Z$  in  $n^{1/2-o(1)}$  different ways to obtain about  $\sqrt{n}$  different values of  $e(G[U \cup Z \cup \{w\}])$  that "fill in" the interval between consecutive values of  $e(G[U \cup Z])$ . Unfortunately, while by construction the degrees  $d_U(w)$ , for  $w \in W$ , are different, it does not follow that the  $d_{U \cup Z}(w)$  are also different. In order to make this approach work, the authors of [21] came up with a way to introduce some limited randomness into the choice of the sets  $Z$ , and with a rather delicate combination of concentration and anticoncentration arguments they were able to show that there are likely to be many different values of  $d_{U \cup Z}(w)$ .

There are two main obstacles that need to be overcome to prove  $|\Phi(G)| = \Omega(n^2)$  with the above strategy. Most obviously, there is a factor of  $n^\gamma$  that must be eliminated. Recall that this factor originates from the upper bound  $O(|W_0|^2/\sqrt{n} + |W_0|n^\gamma)$  on the expected number of pairs of vertices of  $W_0$  which have the same degree into  $U$ . It does not seem that this estimate itself can be improved, but the unwanted factor of  $n^\gamma$  would disappear if we could arrange for  $W_0$  to have size  $\Omega(n^{1/2+\gamma})$  instead of size  $\Theta(\sqrt{n})$ . Unfortunately, if we want  $W_0$  to be asymptotically larger than  $\sqrt{n}$ , it is no longer possible to guarantee that the degrees of vertices in  $W_0$  fall within an interval of length  $O(\sqrt{n})$ ,

and this would cause problems in other parts of the argument. The way we overcome this issue is by allowing two possibilities for the structure of  $W_0$ . Either  $W_0$  is a set of vertices as before, or  $W_0$  is a set of disjoint *pairs* of vertices  $\{x_1, x_2\}$  with similar values of  $d(\{x_1, x_2\}) = d(x_1) + d(x_2)$ . We may then treat these pairs as we would treat single vertices in the above argument, considering sets  $Z$  that are the union of some subset of the pairs in  $W_0$ . Note that there are  $\Theta(n^2)$  pairs of vertices in  $G$ , but only  $O(n)$  possibilities for  $d(\{x_1, x_2\})$ , so this relaxation gives us a lot of flexibility. This idea actually allows us to take  $W_0$  to be of size  $\Omega(n^{3/4})$ , but also introduces some new complications that must be taken care of. In particular, Bukh and Sudakov’s notion of  $(c, \gamma)$ -diversity is not strong enough to deal with pairs of vertices, so in Section 3 we introduce a new notion of  $(\delta, \varepsilon)$ -richness.

The second main obstacle concerns the final part of the argument, where one shows that there are likely to be many different values of  $d_{U \cup Z}(w)$  among the  $w \in W$ . In [21], for this part of the argument the main random set  $U$  had already been fixed, so the only source of randomness was the much smaller set  $Z$ . In this setting, in order to find close to  $\sqrt{n}$  different values of  $d_{U \cup Z}(w)$  it does not merely suffice to consider the variation induced by the random set  $Z$ : one must also take advantage of the separation between different  $d_U(w)$ , and show that this approximately corresponds to separation between the  $d_{U \cup Z}(w)$ . It seems that with this approach there is an unavoidable loss of a logarithmic factor, and it is not clear how to prove a result stronger than  $|\Phi(G)| = \Omega(n^2 / \log n)$ .

In the present paper we take a somewhat different approach, with a “double-exposure” technique. Specifically, we obtain our random set  $U$  as a random subset of about half of the vertices of a larger random set  $U_0$ . Using the ideas sketched above, we can first use the randomness of  $U_0$  to show that there are subsets  $Z \subseteq W_0$  which give  $\Omega(n)$  different values of  $e(G[U_0 \cup Z])$  separated by  $\Omega(\sqrt{n})$  from each other. Then, we can use the randomness of  $U \subseteq U_0$  to show that for most  $Z$  there are  $\Omega(\sqrt{n})$  different values of  $d_{U \cup Z}(w)$ , leading to  $\Omega(\sqrt{n})$  different values of  $e(G[U \cup Z \cup \{w\}])$  closely clustered around  $e(G[U \cup Z])$ . Of course, we also need to show that this second round of randomness did not cause too much damage to the separation we established with the first round: we need to show that the  $e(G[U \cup Z])$  are likely to be well-separated from each other, using the fact that the  $e(G[U_0 \cup Z])$  were chosen to be well-separated from each other. This can be done by taking advantage of the particular structure of the sets  $Z$ , and considering an appropriate notion of what it means for a sequence of values to be “well-separated”.

### 3 Basic tools

In this section we give a number of general results which will be useful in the proof of Theorem 1. Some of these are well-known, and some are new.

First, as mentioned in the introduction, the following lemma is due to Erdős and Szemerédi [17].

**Lemma 2.** *For any  $C$  there exists  $\varepsilon > 0$  such that every  $C$ -Ramsey graph has edge density between  $\varepsilon$  and  $1 - \varepsilon$ .*

Next, we need the notion of *diversity*, introduced by Bukh and Sudakov [9]. Recall from Section 2 that an  $n$ -vertex graph is  $(c, \gamma)$ -diverse if for each vertex  $x \in V$ , we have  $|N(x) \Delta N(y)| < cn$  for at most  $n^\gamma$  vertices  $y \in V$ . Roughly speaking, this means the neighbourhoods of most pairs of vertices are very different. As in [9], we will actually only ever need to take  $\gamma = 1/5$ , so we write “ $c$ -diverse” as shorthand for “ $(c, 1/5)$ -diverse”.

In the same way as [9, 21], the significance of this notion for us is that in a diverse graph, if  $U$  is a random set of vertices, then for most pairs of vertices their degrees into  $U$  are not too strongly correlated. In addition to this basic notion of diversity we also introduce a notion of diversity for *pairs* of vertices. Say an  $n$ -vertex graph is  $(c, \alpha)_2$ -diverse if for each pair of vertices  $\mathbf{x} = \{x_1, x_2\}$  such

that  $\left|N(x_1)\Delta\overline{N(x_2)}\right| \geq \alpha n$ , one cannot find  $n^{1/5}$  other pairs  $\mathbf{y} = \{y_1, y_2\}$ , disjoint to  $\mathbf{x}$  and each other, such that  $|N(\mathbf{x})\Delta N(\mathbf{y})| < cn$  (recall from Section 1.1 the non-standard multiset definitions of  $N(\mathbf{x})$ ,  $N(\mathbf{y})$  and  $N(\mathbf{x})\Delta N(\mathbf{y})$ ).

In this paper, it will be convenient to deduce diversity from a slightly stronger condition. Say an  $n$ -vertex graph is  $(\delta, \varepsilon)$ -rich if for any vertex subset  $W$  with  $|W| \geq \delta n$ , at most  $n^{1/5}$  vertices  $v$  have  $|N(v) \cap W| < \varepsilon|W|$  or  $|\overline{N(v)} \cap W| < \varepsilon|W|$ . We remark that a slightly different definition of richness appeared in the published version of this paper, which was not quite suitable for our application. We thank Mantas Baksys and Xuanang Chen for bringing this to our attention.

**Lemma 3.** *Let  $G$  be a  $(\delta, \varepsilon)$ -rich graph on a set  $V$  of  $n$  vertices, with  $\delta \leq 1/2$ . Then,*

1.  $G$  is  $(\varepsilon/2)$ -diverse;
2.  $G$  is  $(\alpha\varepsilon/2, \alpha)_2$ -diverse for any  $\alpha \geq 2\delta$ ;
3.  $G$  has at most  $n^{1+1/5}$  pairs  $\{x_1, x_2\} \in \binom{V}{2}$  with  $\left|N(x_1)\Delta\overline{N(x_2)}\right| < (\varepsilon/2)n$ .

*Proof.* For the first statement, for each vertex  $x$  either  $|N(x)| \geq n/2$  or  $|\overline{N(x)}| \geq n/2$ . In the former case, for all but at most  $n^{1/5}$  vertices  $y$  we have  $|N(x) \cap \overline{N(y)}| \geq \varepsilon|N(x)| \geq \varepsilon n/2$ , and in the latter case for all but at most  $n^{1/5}$  vertices  $y$  we have  $|\overline{N(x)} \cap N(y)| \geq \varepsilon|\overline{N(x)}| \geq \varepsilon n/2$ . In either case, there are at most  $n^{1/5}$  vertices  $y$  with  $|N(x)\Delta N(y)| < \varepsilon n/2$ , as desired.

For the second statement, note that if  $\left|N(x_1)\Delta\overline{N(x_2)}\right| \geq \alpha n$  then  $|N(x_1) \cap N(x_2)| \geq (\alpha/2)n$  or  $|\overline{N(x_1)} \cap \overline{N(x_2)}| \geq (\alpha/2)n$ . Suppose that there were a pair  $\mathbf{x}$  and a collection  $Y$  of  $n^{1/5}$  pairs contradicting  $(\alpha\varepsilon/2, \alpha)_2$ -diversity, and suppose without loss of generality that  $|N(x_1) \cap N(x_2)| \geq (\alpha/2)n \geq \delta n$ . Then, for each vertex  $y$  in each  $\mathbf{y} \in Y$ ,

$$\left|\overline{N(y)} \cap N(x_1) \cap N(x_2)\right| \leq |N(\mathbf{x})\Delta N(\mathbf{y})| < (\alpha\varepsilon/2)n \leq \varepsilon|N(x_1) \cap N(x_2)|,$$

and the set of all such  $y$  would contradict  $(\delta, \varepsilon)$ -richness.

For the third statement, we will show that for each of the  $n$  choices of  $x_1$  there are at most  $n^{1/5}$  pairs  $\{x_1, x_2\} \in \binom{V}{2}$  with  $\left|N(x_1)\Delta\overline{N(x_2)}\right| < (\varepsilon/2)n$ . Consider any vertex  $x_1$  and suppose without loss of generality that  $|N(x_1)| \geq n/2$ . There are at most  $n^{1/5}$  vertices  $x_2$  with  $|N(x_2) \cap N(x_1)| < \varepsilon n/2$ , and for all other  $x_2$  we have  $\left|N(x_1)\Delta\overline{N(x_2)}\right| \geq |N(x_2) \cap N(x_1)| \geq (\varepsilon/2)n$ .  $\square$

Now, we show that every  $C$ -Ramsey graph contains a rich induced subgraph of linear size. The proof approach is based on a related lemma due to Bukh and Sudakov [9, Lemma 2.2], which in turn uses ideas from [24, 23].

**Lemma 4.** *For any  $C, \delta > 0$ , there exist  $\varepsilon = \varepsilon(C) > 0$  and  $c = c(C, \delta) > 0$  and  $n_0 = n_0(\delta)$  such that if  $n \geq n_0$  then every  $n$ -vertex  $C$ -Ramsey graph contains a  $(\delta, \varepsilon)$ -rich induced subgraph on at least  $cn$  vertices.*

*Proof.* Suppose for the purpose of contradiction that every set of at least  $cn$  vertices fails to induce a  $(\delta, \varepsilon)$ -rich subgraph, for  $c, \varepsilon$  to be determined. For some large  $K = K(C)$  to be determined, we will inductively construct a sequence of induced subgraphs  $G = G[U_0] \supseteq G[U_1] \supseteq \dots \supseteq G[U_K]$  and

disjoint vertex sets  $S_1, \dots, S_K$  such that for all  $i$ ,  $|U_i| \geq (\delta/4)|U_{i-1}|$ ,  $|S_i| = (cn)^{1/5}/2$ ,  $S_i \subseteq U_{i-1}$ , and

$$[d(S_i, S_j) < 4\varepsilon \text{ for all } j > i] \text{ or } [d(S_i, S_j) > 1 - 4\varepsilon \text{ for all } j > i]. \quad (1)$$

This will suffice, as follows. Without loss of generality suppose that the first case of Equation (1) holds for at least half of the choices of  $i$ , and let  $S$  be the union of the corresponding  $S_i$ . Then one can compute  $d(S) < 4\varepsilon + 2/K$ . For sufficiently small  $\varepsilon$ , large  $K$  and large  $n_0$ , this density is too low for  $G[S]$  not to contain a homogeneous subgraph of size  $C \log n$ , by Lemma 2.

Let  $U_0 = V(G)$ . For  $1 \leq i \leq K$  we will construct  $U_i, S_i$ , assuming  $U_0, \dots, U_{i-1}, S_1, \dots, S_{i-1}$  have already been constructed. For  $c \leq (\delta/4)^K$  we have  $|U_{i-1}| \geq cn$ , so by assumption  $U_{i-1}$  contains a set  $W$  of at least  $\delta|U_{i-1}|$  vertices and a set  $Y$  of  $(cn)^{1/5}$  vertices contradicting  $(\delta, \varepsilon)$ -richness. Suppose without loss of generality that  $|N_{U_{i-1}}(v) \cap W| \leq \varepsilon|W|$  for half the vertices  $v \in Y$ , and let  $S_i$  be the corresponding subset of  $Y$ . Then, let  $U = W \setminus S_i$ , so  $|U| \geq |W|/2$ , and let  $U_i \subseteq U$  be the set of vertices  $v \in U$  with  $d(\{v\}, S_i) \leq 4\varepsilon$ . Now, we just need to show  $|U_i| \geq (\delta/4)|U_{i-1}|$ . To this end, first observe that for all  $y \in S_i$  we have  $d(\{y\}, U) = d_U(y)/|U| \leq (\varepsilon|W|)/(|W|/2) = 2\varepsilon$ . Then,

$$4\varepsilon|U \setminus U_i| < \sum_{v \in U \setminus U_i} d(\{v\}, S_i) \leq \frac{e(U, S_i)}{|S_i|} = \frac{|U|}{|S_i|} \sum_{y \in S_i} d(\{y\}, U) \leq 2\varepsilon|U|,$$

implying that  $|U_i| > |U|/2 \geq (\delta/4)|U_{i-1}|$ , as desired.  $\square$

Next, we will use a very slight variation of the Erdős–Littlewood–Offord theorem. Say a random variable is of  $(n, p)$ -Littlewood–Offord type if it can be expressed in the form  $X = a_1\xi_1 + \dots + a_n\xi_n + C$ , where  $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$  and  $C \in \mathbb{Z}$  are fixed and  $\xi_0, \dots, \xi_n$  are independent, identically distributed  $p$ -Bernoulli random variables (taking the value 1 with probability  $p$  and the value 0 with probability  $1 - p$ ). The following variation of the Erdős–Littlewood–Offord theorem follows from, for example, [8, Lemma A.1].

**Lemma 5.** *Suppose  $X$  is of  $(n, p)$ -Littlewood–Offord type, for  $p = \Omega(1)$  and  $1 - p = \Omega(1)$ . Then for any  $x \in \mathbb{Z}$ ,  $\Pr(X = x) = O(1/\sqrt{n})$ .*

Finally, throughout the proof we will use Markov’s inequality, Chebyshev’s inequality, the Chernoff bound, the Azuma–Hoeffding inequality and Turán’s theorem. Statements and proofs of all of these can be found, for example, in [5].

## 4 Proof of Theorem 1

Very broadly, as outlined in Section 2, the basic idea of our proof is similar to the proof in [21]. We will find many induced subgraphs  $G[U]$  with “well-separated” numbers of edges, and we will augment these subgraphs in many different ways. To be precise, Theorem 1 will be an immediate consequence of the following lemma.

**Lemma 6.** *For any  $C$  there is  $c = c(C) > 0$  such that the following holds. For any  $n$ -vertex  $C$ -Ramsey graph  $G$  and any  $m$  satisfying  $cn^2 \leq m \leq 2cn^2$ , there are disjoint subsets  $U, W \subseteq V$  with  $|e(U) - m| = O(n^{3/2})$  and  $|W| = O(\sqrt{n})$  such that*

$$|\{e(U \cup Z) : Z \subseteq W\}| = \Omega(n^{3/2}).$$

*Proof of Theorem 1 given Lemma 6.* For any  $U, W$  as in Lemma 6, we have

$$|e(U \cup W) - e(U)| = O\left(n^{3/2}\right),$$

because  $|W| = O(\sqrt{n})$  and each  $w \in W$  has fewer than  $n$  neighbours in  $U \cup W$ . That is to say, the  $\Omega(n^{3/2})$  values of  $e(U \cup Z)$  are contained in an interval of length  $O(n^{3/2})$  centered at  $e(U)$ . By applying Lemma 6 to  $\Omega(\sqrt{n})$  values of  $m$  each separated by a sufficiently large multiple of  $n^{3/2}$ , we therefore get  $\Omega(n^2)$  different subgraph sizes.  $\square$

The sets  $W$  in Lemma 6 will be comprised of multiple disjoint subsets  $S, T, X$  with different roles. Roughly speaking, given some  $Z \subseteq W$  containing exactly one element of  $X$ , we will be able to increase the number of edges in  $e(U \cup Z)$  by  $\Theta(n)$  by adding an element of  $S$  to  $Z$ , we will be able to increase the number of edges by  $\Theta(\sqrt{n})$  by exchanging an element of  $S$  in  $Z$  with an element of  $T$ , and we will be able to modify  $e(U \cup Z)$  very finely by  $\Theta(\sqrt{n})$  different amounts by making different choices for the single element of  $X$  in  $Z$ . With different combinations of these operations we will be able to obtain  $\Omega(n^{3/2})$  different values of  $e(U \cup Z)$ .

As outlined in Section 2, due to some technical obstacles we were not actually able to construct sets  $U, S, T, X$  that give us control over subgraph sizes in such a simplistic way. Perhaps our most important new idea, which gives us a lot of flexibility, is that we may allow  $S, T, X$  to be sets of disjoint *pairs* of vertices rather than just vertex sets. The following lemma will be a starting point for our construction.

**Lemma 7.** *For any  $C$  there is  $c > 0$  such that the following holds. For any  $n$ -vertex  $C$ -Ramsey graph  $G$  and any  $m$  satisfying  $cn^2 \leq m \leq 2cn^2$ , there is a vertex set  $U_0 \subseteq V$  with  $|e(U_0) - 4m| = O(n^{3/2})$  and sets  $S, T, X$  of size  $\Theta(\sqrt{n})$  such that*

1. *either  $U_0, S, T, X \subseteq V$  are disjoint sets of vertices or  $S, T, X \subseteq \binom{V}{2}$  are sets of disjoint pairs of vertices such that no vertex appears in more than one of  $U_0, S, T, X$ ;*
2. *there is  $d = \Theta(n)$  such that  $d_{U_0}(\mathbf{x}) = d + O(\sqrt{n})$  for each  $\mathbf{x} \in S \cup T \cup X$ ;*
3. *the degrees from  $S$  into  $U_0$  are smaller by  $\Omega(\sqrt{n})$  than the degrees from  $T$  into  $U_0$  (that is,  $\min_{\mathbf{x} \in T} d_{U_0}(\mathbf{x}) - \max_{\mathbf{x} \in S} d_{U_0}(\mathbf{x}) = \Omega(\sqrt{n})$ );*
4. *for each  $\{\mathbf{x}, \mathbf{y}\} \in \binom{X}{2}$ , we have  $|N_{U_0}(\mathbf{x}) \Delta N_{U_0}(\mathbf{y})| = \Omega(n)$ .*

Note that when we write a variable name in bold, it may be a single vertex or a pair of vertices. Also, we emphasise that we are thinking of  $C$  as a fixed constant, so the constants implied by the asymptotic notation in Lemma 7 may depend on  $C$  (but nothing else).

We will prove Lemma 7 in Section 4.1. Without going into too much detail about the proof, the idea is to first apply Lemma 4 to reduce to an induced subgraph with rich neighbourhoods, then use the pigeonhole principle to find a large set  $L$  of either vertices or pairs of vertices with very similar degrees. Then, we choose  $U_0$  randomly and choose  $S, T, X \subseteq L$  based on this random outcome to satisfy the properties in Lemma 7.

Now, consider a  $C$ -Ramsey graph  $G$ , let  $c$  be as in Lemma 7, and consider some  $m$  satisfying  $cn^2 \leq m \leq 2cn^2$ . Apply Lemma 7 to obtain sets  $U_0, S, T, X$ , and let  $c'$  be a constant such that

$$\min_{\mathbf{x} \in T} d_{U_0}(\mathbf{x}) - \max_{\mathbf{x} \in S} d_{U_0}(\mathbf{x}) \geq 8c'\sqrt{n}. \quad (2)$$

(Such a constant exists by the third property of Lemma 7).

Fix an ordering of the elements of  $S$  and of  $T$ , and let  $\mathcal{P}$  be the set of pairs  $(k, i) \in \mathbb{Z}^2$  with  $c'\sqrt{n} \leq k \leq 2c'\sqrt{n}$  and  $0 \leq i \leq c'\sqrt{n}$ . For each  $(k, i) \in \mathcal{P}$ , define the set  $Z_{k,i}$  to contain the vertices of the first  $k-i$  elements from  $S$  and the first  $i$  elements from  $T$ . Note that  $e(Z_{k,0} \cup U_0) - e(Z_{k-1,0} \cup U_0) = \Theta(n)$  for each  $k$ , because  $d_{U_0}(\mathbf{x}) = \Theta(n)$  for each  $\mathbf{x} \in S$  by the second property of Lemma 7. Also note that  $e(Z_{k,i} \cup U_0) - e(Z_{k,i-1} \cup U_0) = \Theta(\sqrt{n})$  for each  $k, i$ , by the second property of Lemma 7, Equation (2) and the fact that  $e(Z_{k,i}) - e(Z_{k,i-1}) \leq 2c'\sqrt{n}$ . Therefore, as  $(k, i)$  varies lexicographically, the  $e(Z_{k,i} \cup U_0)$  comprise  $\Omega(n)$  roughly evenly spaced values.

Now, let  $U$  be a random subset of  $U_0$ , where each vertex is present with probability  $1/2$  independently. We would like some approximation of the spacing described above still to hold for the collection of random values  $e(U_{k,i})$ , where  $U_{k,i} = Z_{k,i} \cup U$ . Also, we want to use the randomness of  $U$  to show that for most  $k, i$ , the  $\mathbf{x} \in X$  have  $\Omega(\sqrt{n})$  different degrees  $d_{U_{k,i}}(\mathbf{x})$  into  $U_{k,i}$ . In Section 4.2 we will prove the following lemma, from which Lemma 6 will easily follow.

**Lemma 8.** *Let  $G$  be a  $C$ -Ramsey graph, let  $S, T, X, d$  be obtained from an application of Lemma 7, and let  $Z_{k,i}, U, U_{k,i}$  be as defined above. Then, there are constants  $M, \beta, Q > 0$  (with  $Q \geq 3\beta$ ) such that for each  $c'\sqrt{n} \leq k \leq 2c'\sqrt{n}$ , the following hold.*

1. *With probability at least 0.99, there is a set  $\mathcal{I}_k$  of  $(1 - \beta/(2M))c'\sqrt{n}$  values of  $i$  with the following property. For each  $i \in \mathcal{I}_k$ , there is a set  $X_{k,i} \subseteq X$  of size  $\Omega(\sqrt{n})$  such that the  $d_{U_{k,i}}(\mathbf{x})$ , for  $\mathbf{x} \in X_{k,i}$ , are distinct, and all lie in the interval between  $d/2 - Q\sqrt{n}$  and  $d/2 + Q\sqrt{n}$ .*
2. *Let  $e_{k,i} = e(Z_{k,i}) + e(Z_{k,i}, U) = e(U_{k,i}) - e(U)$ . With probability at least 0.99,*

$$|e_{k,0} - \mathbb{E}e_{k,0}| \leq Qn.$$

3. *With probability at least  $1/2$ ,  $e_{k,c'\sqrt{n}} - e_{k,0} \geq 3\beta n$ .*
4. *Let  $\Delta_{k,i} = e_{k,i} - e_{k,i-1}$ . With probability at least 0.99,*

$$\sum_{|\Delta_{k,i}| \geq M\sqrt{n}} |\Delta_{k,i}| \leq \beta n.$$

*(That is to say, the “unusually large” increments  $\Delta_{k,i}$  have low total volume).*

Again we emphasise that we are thinking of  $C$  as fixed, so  $M, \beta, Q$  may depend on  $C$ , via  $c'$  and the constants implied by the asymptotic notation in Lemma 7.

Now we can prove Lemma 6 given Lemmas 7 and 8.

*Proof of Lemma 6.* Apply Lemma 7 to obtain  $U_0, S, T, X$ , and define  $\mathcal{P}, U, Z_{k,i}, U_{k,i}$  as above. We will prove the statement of the lemma for  $W$  being the set of all the vertices in  $S \cup T \cup X$ .

For each  $c'\sqrt{n} \leq k \leq 2c'\sqrt{n}$ , with probability at least 0.4, all four parts of Lemma 8 are satisfied. Let  $\mathcal{K}$  be the set of such  $k$ . Then  $\mathbb{E}[c'\sqrt{n} - |\mathcal{K}|] \leq (0.6)c'\sqrt{n}$ , so by Markov's inequality, with probability at least  $1 - 0.6/0.8 = 1/4$  we have  $|\mathcal{K}| \geq (0.2)c'\sqrt{n}$ . Also, note that  $|\mathbb{E}e(U) - m| = O(n^{3/2})$  and there are  $O(n^3)$  pairs of edges in  $G$  whose presence in  $G[U]$  are dependent (this can only occur if they share a vertex). So,  $\text{Var} e(U) = O(n^3)$  and by Chebyshev's inequality,  $|m - e(U)| = O(n^{3/2})$  with probability at least 0.9. Fix an outcome of  $U$  satisfying both these events, which hold together with probability at least 0.15.

For each  $k$ , we have

$$\mathbb{E}e_{k,0} - \mathbb{E}e_{k-1,0} = (e(Z_{k,0} \cup U_0) - e(Z_{k-1,0} \cup U_0))/2 = \Theta(n),$$

since  $\mathbb{E}d_U(\mathbf{x}) = d_{U_0}(\mathbf{x})/2$  for each  $\mathbf{x} \in S$ . Let  $\mathcal{K}' \subseteq \mathcal{K}$  contain every  $q$ th element of  $\mathcal{K}$ , for sufficiently large  $q$  such that the values of  $\mathbb{E}e_{k,0}$ , for  $k \in \mathcal{K}'$ , are separated by at least  $4Qn$ . Then,  $|\mathcal{K}'| = \Theta(\sqrt{n})$ , and by part 2 of Lemma 8, the values of  $e_{k,0}$ , for  $k \in \mathcal{K}'$ , are separated by at least  $2Qn$ .

Next, by part 3 of Lemma 8 we have  $e_{k,c'\sqrt{n}} - e_{k,0} \geq 3\beta n$  for each  $k \in \mathcal{K}'$ . Consider the range of integers between  $e_{k,0}$  and  $e_{k,0} + 3\beta n$ , and within this range consider  $2\beta\sqrt{n}/M$  intervals of length  $M\sqrt{n}$ , each separated by a distance of  $M\sqrt{n}/2 = \Omega(\sqrt{n})$ . By part 4 of Lemma 8, at most  $\beta\sqrt{n}/M$  of these intervals contain no value of  $e_{k,i}$ . Consider a representative  $e_{k,i}$  from  $\beta\sqrt{n}/M$  different intervals, and let  $\mathcal{I}'_k$  be the set of corresponding indices  $i$ . Let  $\mathcal{I}''_k = \mathcal{I}_k \cap \mathcal{I}'_k$ , so that  $|\mathcal{I}''_k| \geq \beta\sqrt{n}/M - \beta c'\sqrt{n}/(2M) \geq \beta\sqrt{n}/(2M)$ . By construction,  $|e_{k,i} - e_{k,0}| \leq 3\beta n$  for each  $i \in \mathcal{I}''_k$ , and the values of  $e_{k,i}$ , for  $i \in \mathcal{I}''_k$ , are separated by  $\Omega(\sqrt{n})$ . We are assuming that  $Q \geq 3\beta$ , so among choices of  $k \in \mathcal{K}'$ ,  $i \in \mathcal{I}''_k$ , we already have a total of  $\Omega(n)$  different values of  $e(U_{k,i})$  separated by  $\Omega(\sqrt{n})$ .

Now, consider every  $q$ th of these values (in increasing order), for sufficiently large  $q$  such that each resulting pair of values are separated by at least  $2Q\sqrt{n}$ . Let  $\mathcal{P}' \subseteq \{(k,i) : k \in \mathcal{K}', i \in \mathcal{I}''_k\}$  be the corresponding set of indices (so  $|\mathcal{P}'| = \Omega(n)$ ). For each  $(k,i) \in \mathcal{P}'$ , we have  $i \in \mathcal{I}_k$ , so by part 1 of Lemma 8, there is  $X_{k,i}$  such that the values  $d_{U_{k,i}}(\mathbf{x})$ , for  $\mathbf{x} \in X_{k,i}$ , are all different, yet are all in a fixed interval of length  $2Q\sqrt{n}$ . Therefore, among choices of  $(k,i) \in \mathcal{P}'$  and  $\mathbf{x} \in X_{k,i}$ , there are  $\Omega(n^{3/2})$  different values of  $e(U_{k,i} \cup \mathbf{x}) = e(U \cup (Z_{k,i} \cup \mathbf{x}))$ , as desired.  $\square$

## 4.1 Proof of Lemma 7

*Proof of Lemma 7.* First, consider  $\varepsilon = \varepsilon(C)$  from Lemma 4 and note that we can assume  $G$  is  $(\delta, \varepsilon)$ -rich, for  $\delta = \varepsilon/4$ . To see this, first apply Lemma 4 to obtain a  $\Omega(n)$ -vertex  $(\delta, \varepsilon)$ -rich induced subgraph  $G[V'] \subseteq V$ . Since  $\log|V'| \geq (1/2)\log n$ ,  $G[V']$  is still  $2C$ -Ramsey, so by tweaking some constants it suffices to find our desired sets  $U_0, S, T, X$  inside  $G[V']$ .

So, we make the aforementioned richness assumption. By Lemma 3 with  $\alpha = \varepsilon/2$ , this means that  $G$  is both  $\varepsilon/2$ -diverse and  $(\varepsilon^2/4, \varepsilon/2)_2$ -diverse, and there are at most  $n^{1+1/5}$  pairs of vertices  $\{x_1, x_2\}$  with  $|N(x_1) \Delta \overline{N}(x_2)| < (\varepsilon/2)n$ . Note that each of the  $\Omega(n^2)$  sums  $d(\mathbf{x}) = d(x_1) + d(x_2)$ , for  $\mathbf{x} = \{x_1, x_2\} \in \binom{V}{2}$ , lie between 0 and  $2n$ , so by the pigeonhole principle there is some  $d'$  and a collection of  $\Omega(n^{3/2})$  pairs  $H \subseteq \binom{V}{2}$  such that  $d(\mathbf{x}) = d' + O(\sqrt{n})$  for all  $\mathbf{x} \in H$ . Interpret  $H$  as a graph on the vertex set  $V$  with  $\Omega(n^{3/2})$  edges, and obtain a further graph  $H'$  by deleting the  $O(n^{1+1/5}) = o(n^{3/2})$  edges  $\{x_1, x_2\}$  with  $|N(x_1) \Delta \overline{N}(x_2)| < (\varepsilon/2)n$ . Now,  $H'$  either has a vertex  $v$  with  $d(v) = \Omega(n^{3/4})$  or it has a matching with  $\Omega(n^{3/4})$  edges. In the former case let  $d'' = d' - d(v)$  and let  $L \subseteq N_H(v)$  be a set of  $\Omega(n^{3/4})$  neighbours of  $v$  in  $H$ . In the latter case let  $d'' = d'$  and let  $L$  be a set of  $\Omega(n^{3/4})$  pairs comprising a matching in  $H'$ . In both cases, for each  $\mathbf{x} \in L$ , we have  $d(\mathbf{x}) = d'' + O(\sqrt{n})$ .

Next, let  $F \subseteq \binom{L}{2}$  be the set of  $\{\mathbf{x}, \mathbf{y}\} \in \binom{L}{2}$  with  $|N(\mathbf{x}) \Delta N(\mathbf{y})| < (\varepsilon^2/4)n$ . By one of our two diversity assumptions, interpreting  $F$  as a graph, it has  $|L| = \Omega(n^{3/4})$  vertices and maximum degree at most  $n^{1/5}$ , so by Turán's theorem it has an independent set  $A$  with size  $\Omega(|L|/n^{1/5}) = \Omega(\sqrt{n})$ . That is to say, for every  $\{\mathbf{x}, \mathbf{y}\} \in \binom{A}{2}$ , we have  $|N(\mathbf{x}) \Delta N(\mathbf{y})| = \Omega(n)$ .

Now, by Lemma 2 and the  $C$ -Ramsey property,  $e(G) \geq 800cn^2$  for some  $c > 0$ . For  $cn^2 \leq m \leq 2cn^2$ , let  $p = \sqrt{4m/e(G)}$  (so  $p = \Omega(1)$  and  $p \leq 0.1$ ), and let  $U_0$  be a random subset of  $V$  obtained by including each element with probability  $p$  independently. We make a few observations.

**Claim.** *The following five events each hold with probability greater than  $4/5$ .*

1.  $|e(U_0) - 4m| = O(n^{3/2})$ ;

2. there is  $Q \subseteq A$  involving no vertices of  $U_0$ , with  $|Q| \geq (2/3)|A|$ ;
3. there is  $R \subseteq A$  with  $|R| \geq (2/3)|A|$  and  $d_{U_0}(\mathbf{x}) = pd'' + O(\sqrt{n})$  for each  $\mathbf{x} \in R$ ;
4.  $|N_{U_0}(\mathbf{x}) \Delta N_{U_0}(\mathbf{y})| = \Omega(n)$  for each  $\{\mathbf{x}, \mathbf{y}\} \in \binom{A}{2}$ ;
5. the equality  $d_{U_0}(\mathbf{x}) = d_{U_0}(\mathbf{y})$  holds for  $O(\sqrt{n})$  pairs  $\{\mathbf{x}, \mathbf{y}\} \in \binom{A}{2}$ .

*Proof of claim.* For the first property, note that  $\mathbb{E}e(G[U_0]) = 4m$  and there are  $O(n^3)$  pairs of edges in  $G$  whose presence in  $G[U_0]$  are dependent (this can only occur if they share a vertex), so  $\text{Var } e(G[U_0]) = O(n^3)$ . The desired result then follows from Chebyshev's inequality, for a sufficiently large constant implicit in " $O(n^{3/2})$ ".

For the second property, note that the size of the subset  $Q \subseteq A$  of elements of  $A$  which contain no vertices of  $U_0$  has mean at least  $(1-p)^2|A|$  and variance  $O(|A|)$ ; since  $1-p \geq 0.9$  the desired result again follows from Chebyshev's inequality.

For the third property, for each  $\mathbf{x} \in A$  we have  $\mathbb{E}d_{U_0}(\mathbf{x}) = pd'' + O(\sqrt{n})$  and  $\text{Var } d_{U_0}(\mathbf{x}) = O(n)$ , so with at probability at least 0.99 we have  $d_{U_0}(\mathbf{x}) = pd'' + O(\sqrt{n})$ . Let  $R$  be the set of  $\mathbf{x}$  satisfying this bound; we have  $\mathbb{E}|A \setminus R| = (0.01)|A|$ , so by Markov's inequality, with probability at least 4/5 we have  $|A \setminus R| \leq (1/3)|A|$ .

For the fourth property, recall that  $|N(\mathbf{x}) \Delta N(\mathbf{y})| = \Omega(n)$  for each  $\{\mathbf{x}, \mathbf{y}\} \in \binom{A}{2}$ . Note that  $N_{U_0}(\mathbf{x}) \Delta N_{U_0}(\mathbf{y}) = (N(\mathbf{x}) \Delta N(\mathbf{y})) \cap U_0$ , so that  $|N_{U_0}(\mathbf{x}) \Delta N_{U_0}(\mathbf{y})|$  has a binomial distribution with parameters  $|N(\mathbf{x}) \Delta N(\mathbf{y})|$  and  $p$ . Then,  $\Pr(|N_{U_0}(\mathbf{x}) \Delta N_{U_0}(\mathbf{y})| < (p/2)|N(\mathbf{x}) \Delta N(\mathbf{y})|) = e^{-\Omega(n)}$  by the Chernoff bound, and the desired result follows from the union bound.

For the fifth property, for  $\{\mathbf{x}, \mathbf{y}\} \in \binom{A}{2}$ , note that the random variable  $d_{U_0}(\mathbf{x}) - d_{U_0}(\mathbf{y})$  is of  $(|N(\mathbf{x}) \Delta N(\mathbf{y})|, p)$ -Littlewood–Offord type. So, recalling that  $|N(\mathbf{x}) \Delta N(\mathbf{y})| = \Omega(n)$ , we have  $\Pr(d_{U_0}(\mathbf{x}) = d_{U_0}(\mathbf{y})) = O(1/\sqrt{n})$ . The expected number of pairs  $\{\mathbf{x}, \mathbf{y}\} \in \binom{A}{2}$  satisfying  $d_{U_0}(\mathbf{x}) = d_{U_0}(\mathbf{y})$  is therefore  $O(\sqrt{n})$ , and the desired result follows from Markov's inequality.  $\square$

Fix an outcome of  $U_0$  satisfying all 5 of the above properties, and arbitrarily divide  $R \cap Q$ , which has size at least  $|A|/3$ , into two subsets  $Y$  and  $X$  of size  $\Omega(\sqrt{n})$ . Consider the graph on the vertex set  $Y$  of all  $\{\mathbf{x}, \mathbf{y}\} \in \binom{Y}{2}$  with  $d_{U_0}(\mathbf{x}) = d_{U_0}(\mathbf{y})$ . This graph has  $O(\sqrt{n})$  edges, so by Turán's theorem it has an independent set  $B$  of size  $\Omega(\sqrt{n})$ . Order the  $\mathbf{x} \in B$  by  $d_{U_0}(\mathbf{x})$ , let  $S$  be the first  $|B|/3$  elements in this ordering and let  $T$  be the last  $|B|/3$  elements. Since each such  $d_{U_0}(\mathbf{x})$  is distinct, this means  $\min_{\mathbf{x} \in T} d_{U_0}(\mathbf{x}) - \max_{\mathbf{x} \in S} d_{U_0}(\mathbf{x}) \geq |B|/3 = \Omega(\sqrt{n})$ . Let  $d = pd''$  and note that  $d = \Omega(n)$ , because otherwise it would be impossible to simultaneously satisfy properties 3 and 4.  $\square$

## 4.2 Proof of Lemma 8

*Proof of Lemma 8.* The constants  $\beta, M, Q$  will be determined in that order, in terms of each other. Therefore it is convenient to prove the four parts of Lemma 8 in a slightly different order than they are stated.

For the third part, note that  $|e(Z_{k,i}) - e(Z_{k,i-1})| \leq 2c'\sqrt{n}$ , and recall Equation (2). We have

$$\mathbb{E}[e_{k,i} - e_{k,i-1}] \geq \frac{1}{2}(e(Z_{k,i}, U_0) - e(Z_{k,i-1}, U_0)) - 2c'\sqrt{n} \geq 2c'\sqrt{n},$$

for each  $i$ . Let  $\beta = 2(c')^2/3$ , so  $\Delta_k := e_{k,c'\sqrt{n}} - e_{k,0}$  has expectation at least  $3\beta n$ . But  $\Delta_k$  is of  $(\Omega(n), 1/2)$ -Littlewood–Offord type and is therefore symmetrically distributed around its expectation. The desired result follows.

For the fourth part, note that  $\Delta_{k,i}$  has mean  $O(\sqrt{n})$  and is affected by 1 or 2 by the addition or removal of an element to/from  $U$ . So, by the Azuma–Hoeffding inequality,  $\Pr(|\Delta_{k,i}| \geq t) =$

$\exp(-\Omega(t^2/n))$ . Now, for any nonnegative integer random variable  $\xi$ , we have  $\mathbb{E}\xi = \sum_{t=1}^{\infty} \Pr(\xi \geq t)$ , so

$$\begin{aligned} \mathbb{E}\left[|\Delta_{k,i}|\mathbb{1}_{|\Delta_{k,i}|\geq M\sqrt{n}}\right] &= \sum_{t=1}^{\infty} \Pr\left(|\Delta_{k,i}|\mathbb{1}_{|\Delta_{k,i}|\geq M\sqrt{n}} \geq t\right) \\ &= M\sqrt{n} \Pr(|\Delta_{k,i}| \geq M\sqrt{n}) + \sum_{t=M\sqrt{n}}^{\infty} \Pr(|\Delta_{k,i}| \geq t) \\ &= M\sqrt{n}e^{-\Omega(M^2)} + \sum_{t=M\sqrt{n}}^{\infty} \exp(-\Omega(t^2/n)) = e^{-\Omega(M^2)}\sqrt{n}, \end{aligned}$$

uniformly over  $M$ . The desired result follows for sufficiently large  $M$ , by linearity of expectation and Markov's inequality.

Now we prove the first part. For each  $k, i$ , and each  $\{\mathbf{x}, \mathbf{y}\} \in \binom{X}{2}$ , the random variable  $d_{U_{k,i}}(\mathbf{x}) - d_{U_{k,i}}(\mathbf{y})$  is of  $(|N_{U_0}(\mathbf{x}) \Delta N_{U_0}(\mathbf{y})|, 1/2)$ -Littlewood–Offord type. So, we have  $\Pr(d_{U_{k,i}}(\mathbf{x}) = d_{U_{k,i}}(\mathbf{y})) = O(1/\sqrt{n})$ . Let  $H_{k,i}$  be the graph of pairs  $\{\mathbf{x}, \mathbf{y}\} \in \binom{X}{2}$  satisfying  $d_{U_{k,i}}(\mathbf{x}) = d_{U_{k,i}}(\mathbf{y})$ , so we have  $\mathbb{E}e(H_{k,i}) = O(\sqrt{n})$ . By Markov's inequality, with probability at least  $1 - \beta/(400M)$  we have  $e(H_{k,i}) = O(\sqrt{n})$ , in which case by Turán's theorem  $H_{k,i}$  has an independent set  $Y_{k,i}$  of size  $2\gamma\sqrt{n}$ , for some constant  $\gamma = \gamma(\beta, M) > 0$ . The expected proportion of values of  $i$  for which this fails to occur is  $\beta/(400M)$ , and by Markov's inequality again, with probability at least 0.995 it fails for only a  $\beta/(2M)$  proportion.

Also, for each  $\mathbf{x} \in X$ , we have  $\mathbb{E}d_U(\mathbf{x}) = d/2 + O(\sqrt{n})$  and  $\text{Var } d_U(\mathbf{x}) = O(n)$ , so by Chebyshev's inequality, for sufficiently large  $Q$  we have  $|d_U(\mathbf{x}) - d/2| \leq Q\sqrt{n}$  with probability at least  $1 - \gamma/200$ . By Markov's inequality, with probability at least 0.995 there is a set  $Y$  with at least  $(1 - \gamma)|X|$  elements of  $X$  satisfying  $d_U(\mathbf{x}) = d/2 + O(\sqrt{n})$ . For sufficiently large  $Q$ , this means that for each  $x \in X$ ,  $d_U(\mathbf{x})$  lies in the interval between  $d/2 - Q\sqrt{n}$  and  $d/2 + Q\sqrt{n}$ .

With probability at least 0.99 both the above events occur, and we can take  $X_{k,i} = Y_{k,i} \cap Y$  for a  $(1 - \beta/(2M))$  proportion of possibilities of  $i$ . This proves the first part of the lemma.

Finally we prove the second part. Note that  $e_{k,0}$  is a translation of  $\sum_{u \in U} dz_{k,0}(u)$ , which is of  $(O(n), 1/2)$ -Littlewood–Offord type, with all coefficients  $O(\sqrt{n})$ . So,  $\text{Var } e_{k,0} = O(n^2)$  and the desired result follows from Chebyshev's inequality for sufficiently large  $Q$  (note that enlarging  $Q$  cannot make the first part fail to hold). □

## 5 Concluding remarks

In this paper we proved that for any fixed  $C$ , if  $G$  is an  $n$ -vertex graph with no homogeneous subgraph on  $C \log n$  vertices, then  $G$  induces subgraphs of  $\Omega(n^2)$  different sizes. This is best possible, but there are a number of other related questions one could ask about Ramsey graphs. For example, as proposed to us by Tuan Tran, we could ask for  $\Omega(n^3)$  induced subgraphs with different numbers of triangles. The methods in this paper might be helpful for this question, but the main obstacle seems to be that one would want a fairly strong anticoncentration inequality for quadratic polynomials in place of Lemma 5.

Actually, we think it would be interesting in general to explore the extent to which anticoncentration phenomena occur in random subsets of Ramsey graphs. For example, consider the following problem. Let  $A$  be the adjacency matrix of an  $O(1)$ -Ramsey graph  $G$  and let  $x \in \{0, 1\}^n$  be a uniformly random 0-1 vector, so that  $x^T A x$  is the number of edges in a uniformly random induced

subgraph of  $G$ . Is it true that  $\Pr(x^T Ax = c) = O(1/n)$  for all  $c \in \mathbb{Z}$ ? This is closely related to a conjecture of Costello ([13, Conjecture 3]), essentially characterising the matrices  $A$  for which this approximately holds.

Another interesting further direction of research would be to consider the situation where larger homogeneous subgraphs are forbidden (see [2, 6, 4, 22] for some examples of theorems of this type). In particular, a natural weakening of an ambitious conjecture of Alon, Krivelevich and Sudakov [4] is that if  $G$  is an  $n$ -vertex graph with no homogeneous subgraph on  $n/4$  vertices, then this is already enough for  $G$  to induce subgraphs of  $\Omega(e(G))$  different sizes.

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