Singularity of sparse random matrices: simple proofs

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Abstract
Consider a random $n \times n$ zero-one matrix with “sparsity” $p$, sampled according to one of the following two models: either every entry is independently taken to be one with probability $p$ (the “Bernoulli” model), or each row is independently uniformly sampled from the set of all length-$n$ zero-one vectors with exactly $pn$ ones (the “combinatorial” model). We give simple proofs of the (essentially best-possible) fact that in both models, if $\min(p, 1-p) \geq (1+\varepsilon) \log n/n$ for any constant $\varepsilon > 0$, then our random matrix is nonsingular with probability $1 - o(1)$. In the Bernoulli model this fact was already well-known, but in the combinatorial model this resolves a conjecture of Aigner-Horev and Person.

1 Introduction

Let $M$ be an $n \times n$ random matrix with i.i.d. Bernoulli($p$) entries (meaning that each entry $M_{ij}$ satisfies $\Pr(M_{ij} = 1) = p$ and $\Pr(M_{ij} = 0) = 1 - p$). It is a famous theorem of Komlós [15, 16] that for $p = 1/2$ a random Bernoulli matrix is asymptotically almost surely nonsingular: that is, $\lim_{n \to \infty} \Pr(M \text{ is singular}) = 0$. Komlós’ theorem can be generalised to sparse random Bernoulli matrices as follows.

**Theorem 1.1.** Fix $\varepsilon > 0$, and let $p = p(n)$ be any function of $n$ satisfying $\min(p, 1 - p) \geq (1 + \varepsilon) \log n/n$. Then for a random $n \times n$ random matrix $M$ with i.i.d. Bernoulli($p$) entries, we have

$$
\lim_{n \to \infty} \Pr(M \text{ is singular}) = 0.
$$

Theorem 1.1 is best-possible, in the sense that if $\min(p, 1 - p) \leq (1 - \varepsilon) \log n/n$, then we actually have $\lim_{n \to \infty} \Pr(M \text{ is singular}) = 1$ (because, for instance, $M$ is likely to have two identical columns). That is to say, $\log n/n$ is a sharp threshold for singularity. It is not clear when Theorem 1.1 first appeared in print, but strengthenings and variations on Theorem 1.1 have been proved by several different authors (see for example [1, 3, 5, 6]).

Next, let $Q$ be an $n \times n$ random matrix with independent rows, where each row is sampled uniformly from the subset of vectors in $\{0, 1\}^n$ having exactly $d$ ones ($Q$ is said to be a random combinatorial matrix). The study of such matrices was initiated by Nguyen [19], who proved that if $d = n/2$ then $Q$ is asymptotically almost surely nonsingular (where $n \to \infty$ along the even integers). Strengthenings of Nguyen’s theorem have been proved by several authors; see for example [2, 10, 12, 13, 23]. Recently, Aigner-Horev and Person [2] conjectured an analogue of Theorem 1.1 for sparse random combinatorial matrices, which we prove in this note.

**Theorem 1.2.** Fix $\varepsilon > 0$, and let $d = d(n)$ be any function of $n$ satisfying $\min(d, n - d) \geq (1 + \varepsilon) \log n$. Then for a $n \times n$ random zero-one matrix $Q$ with independent rows, where each row is chosen uniformly among the vectors with $d$ ones, we have

$$
\lim_{n \to \infty} \Pr(Q \text{ is singular}) \to 0.
$$

Just like Theorem 1.1, Theorem 1.2 is best-possible in the sense that if $\min(d, n - d) \leq (1 - \varepsilon) \log n$, then $\lim_{n \to \infty} \Pr(M \text{ is singular}) = 1$. Theorem 1.2 improves on a result of Aigner-Horev and Person: they proved the same fact under the assumption that $\lim_{n \to \infty} d/(n^{1/2} \log^{3/2} n) = \infty$ (assuming that $d \leq n/2$).

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The structure of this note is as follows. First, in Section 2 we prove a simple and general lemma (Lemma 2.1) which applies to any random matrix with i.i.d. rows. This lemma distills the essence of (a special case of) an argument due to Rudelson and Vershynin [22]. Essentially, it shows that in order to prove Theorem 1.1 and Theorem 1.2, one just needs to prove some relatively crude estimates about the typical structure of the vectors in the left and right kernels of our random matrices.

Then, in Section 3 and Section 4 we show how to use Lemma 2.1 to give simple proofs of Theorem 1.1 and Theorem 1.2. Of course, Theorem 1.1 is not new, but its proof is extremely simple and it serves as a warm-up for Theorem 1.2. It turns out that in order to analyse the typical structure of the vectors in the left and right kernel, we can work over $\mathbb{Z}_q$ for some small integer $q$ (in fact, we can mostly work over $\mathbb{Z}_2$). This idea is not new (see for example, [2, 4, 8, 9, 10, 11, 18, 20, 21]), but the details here are much simpler.

We remark that with a bit more work, the methods in our proofs can also likely be used to prove the conclusions of Theorem 1.1 and Theorem 1.2 under the weaker (and strictly best-possible) assumptions that $\lim_{n \to \infty} (\min(pm, n - pm) - \log n) = \infty$ and $\lim_{n \to \infty} (\min(d, n - d) - \log n) = \infty$. However, in this note we wish to emphasise the simple ideas in our proofs and do not pursue this direction.

Notation. All logarithms are to base $e$. We use common asymptotic notation, as follows. For real-valued functions $f(n)$ and $g(n)$, we write $f = O(g)$ to mean that there is some constant $C > 0$ such that $|f| \leq Cg$. If $g$ is nonnegative, we write $f = \Omega(g)$ to mean that there is $c > 0$ such that $f \geq cg$ for sufficiently large $n$. We write $f = o(g)$ to mean that $f(n)/g(n) \to 0$ as $n \to \infty$.

Acknowledgements. We would like to thank Elad Aigner-Horev, Yury Person, and the anonymous referee, for helpful comments and suggestions.

## 2 A general lemma

In this section we prove a (very simple) lemma which will give us a proof scheme for both Theorem 1.1 and Theorem 1.2. For a vector $x$, let $\text{supp}(x)$ (the support of $x$) be the set of indices $i$ such that $x_i \neq 0$.

**Lemma 2.1.** Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$ be a random matrix with i.i.d. rows $R_1, \ldots, R_n$. Let $\mathcal{P} \subseteq \mathbb{F}^n$ be any property of vectors in $\mathbb{F}^n$. Then for any $t \in \mathbb{R}$, the probability that $A$ is singular is upper-bounded by

\[
\Pr(x^TA = 0 \text{ for some nonzero } x \in \mathbb{F}^n \text{ with } |\text{supp}(x)| < t) + \frac{n}{t} \Pr(\text{there is nonzero } x \notin \mathcal{P} \text{ such that } x \cdot R_i = 0 \text{ for all } i = 1, \ldots, n - 1) + \frac{n}{t} \sup_{x \in \mathcal{P}} \Pr(x \cdot R_n = 0) \tag{2.1, 2.2, 2.3}
\]

**Proof.** Note that $A$ is singular if and only if there is a nonzero $x \in \mathbb{F}^n$ satisfying $x^TA = 0$. Let $\mathcal{E}_i$ be the event that $R_i \in \text{span}\{R_1, \ldots, R_{i-1}, R_{i+1}, \ldots, R_n\}$, and let $X$ be the number of $i$ for which $\mathcal{E}_i$ holds. Then by Markov’s inequality and the assumption that the rows $R_1, \ldots, R_n$ are i.i.d., we have

\[
\Pr(x^TM = 0 \text{ for some } x \text{ with } |\text{supp}(x)| \geq t) \leq \frac{\mathbb{E}X}{t} = \frac{n}{t} \Pr(\mathcal{E}_n).
\]

It now suffices to show that $\frac{n}{t} \Pr(\mathcal{E}_n)$ is upper-bounded by the sum of the terms (2.2) and (2.3). Note that we can always choose a nonzero vector $x \in \mathbb{F}^n$ with $x \cdot R_i = 0$ for $i = 1, \ldots, n - 1$. We interpret $x$ as a random vector depending on $R_1, \ldots, R_{n-1}$ (but not $R_n$). If the event $\mathcal{E}_n$ occurs, we must have $x \cdot R_n = 0$, so

\[
\frac{n}{t} \Pr(\mathcal{E}_n) \leq \frac{n}{t} \Pr(x \notin \mathcal{P}) + \frac{n}{t} \Pr(x \cdot R_n = 0 \mid x \in \mathcal{P}).
\]

Then $\frac{n}{t} \Pr(x \notin \mathcal{P})$ is upper-bounded by the expression in (2.2), and, since $x$ and $R_n$ are independent, $\frac{n}{t} \Pr(x \cdot R_n = 0 \mid x \in \mathcal{P})$ is upper-bounded by the expression in (2.3). \qed
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Let us fix $0 < \varepsilon < 1$. We will take $t = cn$ for some small constant $c$ (depending on $\varepsilon$), and let $P$ be the property \( \{ x \in \mathbb{Q}^n : |\text{supp}(x)| \geq t \} \). All we need to do is to show that the three terms (2.1), (2.2) and (2.3) in Lemma 2.1 are each of the form $o(1)$. The following lemma is the main part of the proof.

Lemma 3.1. Let $R_1, \ldots, R_{n-1}$ be the first $n-1$ rows of a random Bernoulli($p$) matrix, with $\min(p, 1-p) \geq (1 + \varepsilon) \log n/n$. There is $c > 0$ (depending only on $\varepsilon$) such that with probability $1 - o(1)$, no nonzero vector $x \in \mathbb{Q}^n$ with $|\text{supp}(x)| < cn$ satisfies $R_i \cdot x = 0$ for all $i = 1, \ldots, n - 1$.

Proof. If such a vector $x$ were to exist, we would be able to multiply by an integer and then divide by a power of two to obtain a vector $v \in \mathbb{Z}^n$ with at least one odd entry also satisfying $|\text{supp}(v)| < cn$ and $R_i \cdot v = 0$ for $i = 1, \ldots, n - 1$. Interpreting $v$ as a vector in $\mathbb{Z}_2^n$, we would have $R_i \cdot v \equiv 0 \pmod{2}$ for $i = 1, \ldots, n - 1$ and furthermore $v \in \mathbb{Z}_2^n$ would be a nonzero vector consisting of less than $cn$ ones. We show that such a vector $v$ is unlikely to exist (working over $\mathbb{Z}_2$ discretises the problem, so that we may use a union bound).

Let $p^* = \min(p, 1-p) \geq (1 + \varepsilon) \log n/n$. Consider any $v \in \{0, 1\}^n$ with $|\text{supp}(v)| = s$. Then $R_i \cdot v$ for $i = 1, \ldots, n - 1$ are i.i.d. Binomial($s, p$) random variables. Let $P_{s, p}$ denote the probability that a Binomial($s, p$) random variable is even. We observe

\[
P_{s, p} = \frac{1}{2} \left( \sum_{i=0}^{s} \binom{s}{i} p^i (1-p)^{s-i} + \sum_{i=0}^{s} \binom{s}{i} (-1)^i p^i (1-p)^{s-i} \right) = \frac{1}{2} + \frac{(1 - 2p)^{s}}{2} \leq \frac{1}{2} + \frac{(1 - 2p^*)^{s}}{2}.
\]

Then, using the fact that $e^{-t} = 1 - t + O(t^2)$ for $t = o(1)$, we deduce

\[
P_{s, p} \leq \begin{cases} 
e^{-s\log(1/s \cdot p^*)} & \text{if } sp^* = o(1), \\ e^{-\Omega(1)} & \text{if } sp^* = \Omega(1). \end{cases}
\]

Taking $r = \delta/p^*$ for sufficiently small $\delta$ (relative to $\varepsilon$), and recalling that $p^* \geq (1 + \varepsilon) \log n/n$, the probability that there exists nonzero $v \in \mathbb{Z}_2^n$ with $|\text{supp}(v)| < cn$ and $R_i \cdot v \equiv 0 \pmod{2}$ for all $i = 1, \ldots, n - 1$ is at most

\[
\sum_{s=1}^{cn} \binom{n}{s} p_{s, p}^{n-1} \leq \sum_{s=1}^{r} \frac{n^s \log n - (1-\varepsilon/2) s p^*}{s+1} + \sum_{s=r+1}^{cn} e^{s \log(n/s) + 1 - \Omega(n)} \leq \sum_{s=1}^{\infty} n^{-s \varepsilon/3} + e^{c_n (s/n) \log(n/s) + 1 - \Omega(1)} = o(1),
\]

provided $c$ is sufficiently small (relative to $\delta$).

Taking $c$ as in Lemma 3.1, we immediately see that the term (2.2) is of the form $o(1)$. Observing that the rows and columns of $M$ have the same distribution, and that the event $x^T M = 0$ is simply the event that $x \cdot C_i = 0$ for each column $C_i$ of $M$, it also follows from Lemma 3.1 that the term (2.1) is of the form $o(1)$. Finally, the following straightforward generalisation of the well-known Erdős–Littlewood–Offord theorem shows that the term (2.3) is of the form $o(1)$, which completes the proof of Theorem 1.1. This lemma is the only nontrivial ingredient in the proof of Theorem 1.1. It appears as [5, Lemma 8.2], but it can also be quite straightforwardly deduced from the Erdős–Littlewood–Offord theorem itself.

Lemma 3.2. Consider a (non-random) vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and let $\xi_1, \ldots, \xi_n$ be i.i.d. Bernoulli($p$) random variables, and let $p^* = \min(p, 1 - p)$. Then

\[
\max_{a \in \mathbb{R}} \Pr(x_1 \xi_1 + \cdots + x_n \xi_n = a) = O\left(\frac{1}{|\text{supp}(x)| p^*}\right).
\]
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Let us again fix $0 < \varepsilon < 1$. The proof of Theorem 1.2 proceeds in almost exactly the same way as the proof of Theorem 1.1, but there are three significant complications. First, since the entries are no longer independent, the calculations become somewhat more technical. Second, the rows and columns of $Q$ have different distributions, so we need two versions of Lemma 3.1: one for vectors in the left kernel and one for vectors in the right kernel. Third, the fact that each row has exactly $d$ ones means that we are not quite as free to do computations over $\mathbb{Z}_2$ (for example, if $d$ is even and $v$ is the all-ones vector then we always have $Qv = 0$ over $\mathbb{Z}_2$). For certain parts of the argument we will instead work over $\mathbb{Z}_{d-1}$.

Before we start the proof, the following lemma will allow us to restrict our attention to the case where $d \leq n/2$, which will be convenient.

**Lemma 4.1.** Let $Q \in \mathbb{R}^{n \times n}$ be a matrix whose every row has sum $d$, for some $d \notin \{0, n\}$. Let $J$ be the $n \times n$ all-ones matrix. Then $Q$ is singular if and only if $J - Q$ is singular.

**Proof.** Note that the all-ones vector $1$ is in the column space of $Q$ (since the sum of all columns of $Q$ equals $d1$). Hence every column of $J - Q$ is in the column space of $Q$. Therefore, if $Q$ is singular, then $J - Q$ is singular as well. The opposite implication can be proved the same way. \hfill \Box

In the rest of the section we prove Theorem 1.2 under the assumption that $(1 + \varepsilon) \log n \leq d \leq n/2$ (note that if $Q$ is a uniformly random zero-one matrix with every row having exactly $d$ ones, then $J - Q$ is a uniformly random zero-one matrix with every row having exactly $n - d$ ones).

The first ingredient we will need is an analogue of Lemma 3.2 for “combinatorial” random vectors. In addition to the notion of the support of a vector, we define a fibre of a vector to be a set of all indices whose entries are equal to a particular value.

**Lemma 4.2.** Let $0 \leq d \leq n/2$, and consider a (non-random) vector $x \in \mathbb{R}^n$ whose largest fibre has size $n - s$, and let $\gamma \in \{0, 1\}^n$ be a random zero-one vector with exactly $d$ ones. Then

$$\max_{a \in \mathbb{R}} \Pr(x \cdot \gamma = a) = O\left(\sqrt{n/(sd)}\right).$$

We deduce Lemma 4.2 from the $p = 1/2$ case of Lemma 3.2 (that is, from the Erdős–Littlewood–Offord theorem [7]).

**Proof.** The case $p = 1/2$ is treated in [17, Proposition 4.10]; this proof proceeds along similar lines. Let $p = d/n \leq 1/2$. We realise the distribution of $\gamma$ as follows. First choose $d = pn$ random disjoint pairs $(i_1, j_1), \ldots, (i_{pn}, j_{pn}) \in \{1, \ldots, n\}^2$ (each having distinct entries), and then determine the 1-entries in $\gamma$ by randomly choosing one element from each pair.

We first claim that with probability $1 - e^{-\Omega(sp)}$, at least $\Omega(sp)$ of our pairs $(i, j)$ have $x_i \neq x_j$ (we say such a pair is good). To see this, let $I$ be a union of fibres of $x$, chosen such that $|I| \geq n/3$ and $n - |I| \geq s/3$ (if $s \leq 2n/3$ we can simply take $I$ to be the largest fibre of $x$, and otherwise we can greedily add fibres to $I$ until $|I| \geq n/3$). To prove our claim, we will prove that in fact with the desired probability there are $\Omega(sp)$ different $\ell$ for which $i_\ell \notin I$ and $j_\ell \in I$.

Let $f = [pn/6]$ and let $S$ be the set of $\ell \leq f$ for which $i_\ell \notin I$. So, $|S|$ has a hypergeometric distribution with mean $(n - |I|)f/n = \Omega(sp)$, and by a Chernoff bound (see for example [14, Theorem 2.10]), we have $|S| = \Omega(sp)$ with probability $1 - e^{-\Omega(sp)}$. Condition on such an outcome of $i_1, \ldots, i_f$. Next, let $T$ be the set of $\ell \in S$ for which $j_\ell \in I$. Then, conditionally, $|T|$ has a hypergeometric distribution with mean at least $(|I| - f)|S|/n = \Omega(sp)$, so again using a Chernoff bound we have $|T| = \Omega(sp)$ with probability $1 - e^{-\Omega(sp)}$, as claimed.

Now, condition on an outcome of our random pairs such that at least $\Omega(sp)$ of them are good. Let $\xi_\ell$ be the indicator random variable for the event that $i_\ell$ is chosen from the pair $(i_\ell, j_\ell)$, so $\xi_1, \ldots, \xi_{pn}$ are i.i.d. Bernoulli($1/2$) random variables, and $x \cdot \gamma = a$ if and only if

$$(x_{i_1} - x_{j_1})\xi_1 + \cdots + (x_{i_{pn}} - x_{j_{pn}})\xi_1 = a - x_{j_1} - \cdots - x_{j_{pn}}.$$
Under our conditioning, $\Omega(sp)$ of the $x_{ij} - x_{ji}$ are nonzero, so by Lemma 3.2 with $p = 1/2$, conditionally we have $\Pr(x \cdot \gamma = a) \leq O(1/\sqrt{sp})$. We deduce that unconditionally

$$\Pr(x \cdot \gamma = 0) \leq e^{-(sp)} + O(1/\sqrt{sp}) = O(1/\sqrt{sp}) = O(\sqrt{n/(sd^2)}),$$

as desired. \hfill \Box

The proof of Theorem 1.2 then reduces to the following two lemmas. Indeed, for a constant $c > 0$ (depending on $\varepsilon$) satisfying the statements in Lemmas 4.3 and 4.4, we can take $t = cn/\log d$, and

$$\mathcal{P} = \{x \in \mathbb{Q}^n : x \text{ has largest fibre of size at most } (1 - c/\log d)n\}.$$

We can then apply Lemma 2.1. By Lemma 4.3, the term (2.1) is bounded by $o(1)$, by Lemma 4.4 the term (2.2) is bounded by $(n/t) \cdot n^{-\Omega(1)} = (\log d/c) \cdot n^{-\Omega(1)} = o(1)$, and by Lemma 4.2 the term (2.3) is bounded by $(n/t) \cdot O\left(\sqrt{n \log d/(cnd^2)}\right) = O((\log^{3/2} d/\sqrt{d}) = o(1)$.

**Lemma 4.3.** Let $Q$ be a random combinatorial matrix (with $d$ ones in each row), with $(1 + \varepsilon) \log n \leq d \leq n/2$. There is $c > 0$ (depending only on $\varepsilon$) such that with probability $1 - o(1)$, there is no nonzero vector $x \in \mathbb{Q}^n$ with $|\text{supp}(x)| < cn/\log d$ and $x^T Q = 0$.

**Lemma 4.4.** Let $R_1, \ldots, R_{n-1}$ be the first $n - 1$ rows of a random combinatorial matrix (with $d$ ones in each row), with $(1 + \varepsilon) \log n \leq d \leq n/2$. There is $c > 0$ (depending only on $\varepsilon$) such that with probability $1 - n^{-\Omega(1)}$, every nonzero $x \in \mathbb{Q}^n$ satisfying $R_i \cdot x = 0$ for all $i = 1, \ldots, n - 1$ has largest fibre of size at most $(1 - c/\log d)n$.

**Proof of Lemma 4.3.** As in Lemma 3.1, it suffices to work over $\mathbb{Z}_2$. Let $C_1, \ldots, C_n$ be the columns of $Q$, consider any $v \in \mathbb{Z}_2^n$ with $|\text{supp}(v)| = s$, and let $\mathcal{E}_v$ be the event that $C_i \cdot v = 0 \mod 2$ for $i = 1, \ldots, n$. Note that $\mathcal{E}_v$ only depends on the submatrix $Q_v$ of $Q$ containing only those rows $j$ with $v_j = 1$ (and $\mathcal{E}_v$ is precisely the event that every column of $Q_v$ has an even sum).

Let $p = d/n \leq 1/2$, let $M_v$ be a random $s \times n$ matrix with i.i.d. Bernoulli($p$) entries, and let $\mathcal{E}'_v$ be the event that every column in $M_v$ has an even sum. Note that $M_v$ is very similar to $Q_v$, so the probability of $\mathcal{E}_v$ is very similar to the probability of $\mathcal{E}'_v$. Indeed, writing $R_1, \ldots, R_s$ and $R_1', \ldots, R_s'$ for the rows of $Q_v$ and $M_v$ respectively, and writing $s_j = |\text{supp}(R_j')|$, for each $j$ we have $s_j \sim \text{Binomial}(n, p)$, so an elementary computation using Stirling’s formula shows that $\Pr(s_j = d) = \Omega(1/\sqrt{d}) = e^{-O(\log d)}$. Hence

$$\Pr(\mathcal{E}_v) = \Pr(\mathcal{E}'_v) \cdot \Pr(s_j = d \text{ for all } j) \leq \Pr(\mathcal{E}'_v) / \Pr(s_j = d \text{ for all } j) = e^{O(s \log d)} \Pr(\mathcal{E}'_v) = e^{O(s \log mn)} \Pr(\mathcal{E}'_v).$$

Recalling the quantity $P_{s,p}$ from the proof of Lemma 3.1, we have

$$\Pr(\mathcal{E}'_v) = P_{s,p}^{n} = \begin{cases} e^{-(1+o(1))spn} & \text{if } sp = o(1), \\ e^{-\Omega(n)} & \text{if } sp = \Omega(1), \end{cases}$$

so if $s \leq cn/\log d = cn/\log (mn)$ for small $c > 0$, then we also have

$$\Pr(\mathcal{E}_v) \leq \begin{cases} e^{-(1+o(1))spn} & \text{if } sp = o(1), \\ e^{-\Omega(n)} & \text{if } sp = \Omega(1). \end{cases}$$

Let $P_s = \Pr(\mathcal{E}_v)$ (which only depends on $s$). We can now conclude the proof in exactly the same way as in Lemma 3.1. Taking $r = \delta/p$ for sufficiently small $\delta$ (relative to $\varepsilon$), the probability that there exists nonzero $v \in \mathbb{Z}_2^n$ with $|\text{supp}(v)| < cn/\log d$ and $C_i \cdot v \equiv 0 \mod 2$ for all $i = 1, \ldots, n$ is at most

$$\sum_{s=1}^{cn/\log d} \binom{n}{s} P_s \leq \sum_{s=1}^{r} e^{s \log n - (1-\varepsilon/3)spn} + \sum_{s=r+1}^{cn/\log d} e^{s(\log(n/s)+1) - \Omega(n)}$$

$$\leq \sum_{s=1}^{\infty} n^{-se/3} + \sum_{s=1}^{cn/\log d} e^{n((s/n)\log(n/s)+1) - \Omega(1)} = o(1),$$

provided $c$ is sufficiently small (relative to $\delta$). \hfill \Box
We will deduce Lemma 4.4 from the following lemma.

**Lemma 4.5.** Suppose \( p \leq 1/2 \) and \( pn \to \infty \), and let \( \gamma \in \{0,1\}^n \) be a random vector with exactly \( pn \) ones. Let \( q \geq 2 \) be an integer and consider a (non-random) vector \( v \in \mathbb{Z}_q^n \) whose largest fibre has size \( n-s \). Then for any \( a \in \mathbb{Z}_q \) we have \( \Pr(v \cdot \gamma \equiv a \mod q) \leq P_{p,n,s} \) for some \( P_{p,n,s} \) (only depending on \( p, n \) and \( s \)) satisfying

\[
P_{p,n,s} = \begin{cases} e^{-\Omega(1)} & \text{when } sp = \Omega(1), \\
e^{-(1-o(1))sp} & \text{when } sp = o(1) \\
\end{cases}
\]

**Proof.** As in the proof of Lemma 4.2, we realise the distribution of \( \gamma \) by first choosing \( pn \) random disjoint pairs \((i_1,j_1), \ldots, (i_{pn}, j_{pn}) \in \{1, \ldots, n\}^2\), and then randomly choosing one element from each pair to comprise the 1-entries of \( \gamma \).

Let \( E \) be the event that \( v_i \neq v_j \) for at least one of our random pairs \((i,j)\). Then \( \Pr(v \cdot \gamma \equiv a \mod q) \mid E \leq 1/2 \), and therefore \( \Pr(v \cdot \gamma \equiv a \mod q) \leq 1 - \Pr(E)/2 \). So, it actually suffices to prove that

\[
\Pr(E) \geq \Omega(1)
\]

Let \( F \) be the event that \( i_k \in \mathcal{I} \) for all \( k = 1, \ldots, pn \). We have

\[
\Pr(E \mid F) \geq 1 - (1 - s/n)^{pn} = \begin{cases} \Omega(1) & \text{when } sp = \Omega(1), \\
(1 - o(1))sp & \text{when } sp = o(1), \\
\end{cases}
\]

and

\[
\Pr(E \mid \overline{F}) \geq (n-s-pn)/(n-pn) = \begin{cases} \Omega(1) & \text{when } sp = \Omega(1), \\
1 - o(1) & \text{when } sp = o(1). \\
\end{cases}
\]

This already implies that if \( sp = \Omega(1) \), then \( \Pr(E) = \Omega(1) \) as desired. If \( sp = o(1) \) then \( \Pr(F) \leq (1 - s/n)^{pn} = 1 - (1 + o(1))sp \), so

\[
\Pr(E) = \Pr(F) \Pr(E \mid F) + \Pr(\overline{F}) \Pr(E \mid \overline{F}) \geq (2 - o(1))sp,
\]

as desired. \( \square \)

**Proof of Lemma 4.4.** Let \( q = d-1 \). It suffices to prove that with probability \( 1 - o(1) \) there is no nonconstant “bad” vector \( v \in \mathbb{Z}_q^n \) whose largest fibre has size at least \( (1 - c/\log q)n \) and which satisfies \( R_i \cdot v \equiv 0 \mod q \) for all \( i = 1, \ldots, n-1 \). (Note that by the choice of \( q \), if \( v \in \mathbb{Z}_q^n \) is constant and nonzero, then it is impossible to have \( v \cdot R_1 = 0 \).

Let \( p = d/n \), consider any \( v \in \mathbb{Z}_q^n \) whose largest fibre has size \( n-s \), and consider any \( i \in \{1, \ldots, n-1\} \). Then \( R_i \cdot v \) is of the form in Lemma 4.5, so taking \( r = \delta/p \) for sufficiently small \( \delta \) (relative to \( \varepsilon \)), the probability that such a bad vector exists is at most

\[
\sum_{s=1}^{c'n/\log q} \binom{n}{s} q^{s+1} p^{n-1}_{p,n,s} \leq \sum_{s=1}^{c'n/\log q} e^{s \log n + (s+1)2\sqrt{sp} - (1-\varepsilon/3)sp} + \sum_{s=r+1}^{c'n/\log q} e^{s \log (n/s) + 1 \log (n/s) + 1 - \Omega(1)}
\]

\[
\leq \sum_{s=1}^{\infty} n^{-sc/3} + \sum_{s=1}^{c'n/\log q} e^{n(s/n)(\log (n/s) + 1 - \Omega(1))} = n^{-\Omega(1)},
\]

provided \( c' > 0 \) is sufficiently small (relative to \( \delta \)) and \( n \) is sufficiently large. \( \square \)