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Local boundedness of local suitable weak solutions of the Navier–Stokes equations

Masterarbeit

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If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.

— John von Neumann
Mathematician (1903 – 1957)

Abstract

In the present work we study a subclass of weak solutions of the 3D Navier–Stokes equations, the so-called local suitable weak solutions due to Wolf [31]. This notion of solution does not require a globally defined pressure field. Instead, based on an existence result for the forced steady Stokes system we work with a local pressure decomposition of the form $p = \partial_t p_h + p_0$, where p_h is harmonic. This particular structure is also reflected in the corresponding local energy inequality for local suitable weak solutions.

A very important feature of this construction is that decay estimates for the pressure fields can be expressed in terms of the velocity field alone. We make use of this in order to study the (interior) local regularity of local suitable weak solutions. Our main result in this context is a local L^∞-bound for the velocity field in terms of the rescaled maximal kinetic energy, i.e. the rescaled $L^{2,\infty}$-norm of the velocity field over some parabolic cylinder in space-time. An analogous estimate also holds for the spatial derivatives of the velocity field.

Contents

1 Introduction

In this work, we discuss a result related to the local ε-regularity theory for the 3D Navier–Stokes equations

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) = -\nabla p, \\ \operatorname{div} \mathbf{u} = 0. \end{cases} \tag{1.1}$$

More precisely, we will study the 3D Navier–Stokes equations in a space-time cylinder denoted by $Q = \Omega \times (a,b)$, where Ω always represents some domain in $\mathbb{R}^3$. In the system (1.1), we denote by $\mathbf{u} \colon \Omega \times (a,b) \to \mathbb{R}^3$ the unknown velocity field and by $p \colon \Omega \times (a,b) \to \mathbb{R}$ the unknown scalar pressure field.

Before we overview some notions of solutions attached to the Navier–Stokes equations, let us introduce related important function spaces. In the following, we denote by $C_c^\infty(\Omega)$ the space of all smooth and compactly supported functions and by $C_{c,\sigma}^\infty(\Omega)$ the subspace of all solenoidal fields, i.e. $C_{c,\sigma}^\infty(\Omega) = \left\{\phi \in C_c^\infty(\Omega) \colon \operatorname{div} \phi = 0\right\}$. Furthermore, we write $L_\sigma^2(\Omega)$ (resp. $W_{0,\sigma}^{1,2}(\Omega)$) for the closure of $C_{c,\sigma}^\infty(\Omega)$ in $L^2(\Omega)$ (resp. $W_0^{1,2}(\Omega)$), where $L^p(\Omega)$, $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ are the usual Lebesgue and Sobolev spaces. We also use $L_0^p(\Omega)$ to denote the subspace of all L^p-functions which integrate to zero over Ω. Finally, for a real Banach space X we use $L^p(a,b;X)$ to denote the Lebesgue–Bochner space. In case $X = L^s(\Omega)$ we denote the associated norm by $\|\cdot\|_{L^{s,p}(Q)}$.

1.1 Notions of solutions

In this work, we will solely present and prove interior regularity results. In particular, we will typically not be concerned with any kind of initial and boundary data. Furthermore, our method works without the need of a globally defined pressure field. Therefore, we decided to give the following definition for a weak solution of the Navier–Stokes equations.

Definition 1.1. Let $\mathbf{u} \colon Q \to \mathbb{R}^3$ be a measurable function. We say that $\mathbf{u}$ is a weak solution of the Navier–Stokes equations (1.1) in the space-time cylinder $Q = \Omega \times (a,b)$ if

$$\mathbf{u} \in L^\infty\big(a,b;L^2(\Omega;\mathbb{R}^3)\big) \cap L^2\big(a,b;W^{1,2}(\Omega;\mathbb{R}^3)\big), \tag{1.2}$$

the equation $\operatorname{div} \mathbf{u} = 0$ holds in the sense of distributions and, for all test functions $\boldsymbol{\varphi} \in C_c^1\big((a,b);C_{c,\sigma}^\infty(\Omega;\mathbb{R}^3)\big)$, it holds

$$-\int_Q \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} \,\mathrm{d}x\mathrm{d}t + \int_Q \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} \,\mathrm{d}x\mathrm{d}t - \int_Q (\mathbf{u} \otimes \mathbf{u}) : \nabla \boldsymbol{\varphi} \,\mathrm{d}x\mathrm{d}t = 0. \tag{1.3}$$ ◇

Remark 1.2. The verification of existence of weak solutions is due to Leray and Hopf [14, 11]. In fact, the constructed solutions exist globally in time and possess more properties than our definition initially requires, e.g. an energy inequality in terms of the initial kinetic energy. The corresponding notion of weak solution is then known under the name of *Leray–Hopf solution* (cf. [6]). ♢

When it comes to local and partial regularity results for the Navier–Stokes equations, the notion of a suitable weak solution is commonly used throughout the literature. Inspired by ideas and results due to Scheffer [18, 19], Caffarelli et al. eventually established it in their celebrated paper [1]. Here, we present a version which is due to Lin [15].

Definition 1.3. Let $\mathbf{u}\colon Q \to \mathbb{R}^3$ and $p\colon Q \to \mathbb{R}$ be measurable functions. We call the pair $(\mathbf{u}, p)$ a suitable weak solution of the Navier–Stokes equations (1.1) in the space-time cylinder $Q = \Omega \times (a, b)$ if

$$\mathbf{u} \in L^\infty\big(a, b; L^2(\Omega; \mathbb{R}^3)\big) \cap L^2\big(a, b; W^{1,2}(\Omega; \mathbb{R}^3)\big), \tag{1.4}$$

$$p \in L^{\frac{3}{2}}(Q), \tag{1.5}$$

the system (1.1) is satisfied in the sense of distributions and the following generalized energy inequality holds

$$\begin{aligned} &\int_\Omega |\mathbf{u}(t)|^2 \phi(t)\, \mathrm{d}x + 2\int_a^t \int_\Omega |\nabla \mathbf{u}|^2 \phi\, \mathrm{d}x\mathrm{d}s \\ &\leq \int_a^t \int_\Omega \mathbf{u}^2(\partial_t \phi + \Delta\phi)\, \mathrm{d}x\mathrm{d}s + \int_a^t \int_\Omega \big(|\mathbf{u}|^2 + 2p\big)\mathbf{u} \cdot \nabla\phi\, \mathrm{d}x\mathrm{d}s. \end{aligned} \tag{1.6}$$

for a.e. $t \in (a, b)$ and every non-negative test function $\phi \in C_c^\infty(Q)$. ♢

The occurring integrability condition for the pressure does not constitute the only possibility. For instance, in [1] the authors work with the condition $p \in L^{5/4}(Q)$. A clarification on this matter is given in [13]. Let us say at least that the answer is directly connected to the question, in which sense one can associate a pressure field to a weak solution of the Navier–Stokes equations, if this is possible at all. (Remember that the definition of a weak solution itself gives no information on the pressure.)

In the present work, roughly speaking we are working with a hybrid of weak and suitable weak solutions. This stems from the fact that we want to obtain results which do not require any assumptions on the domain. Therefore, the notion of a suitable weak solution is too strong in the sense that in this situation there may be no globally defined pressure as it is required in the definition (cf. [5]). Instead, we will work with a local pressure decomposition of the type $p = \partial_t p_h + p_0$. This construction can already be worked out within the class of weak solutions.

On the other side, the notion of a weak solution as we defined it here is too weak in the sense that it gives us no sufficient control of the local energy (cf. the corresponding discussion in [31]). But this is exactly what an energy inequality of the form (1.6) provides us. Consequently, it is natural to work out a localized version of (1.6) which then also reflects the local pressure decomposition mentioned before.

Note that in our context a local pressure representation encoded in terms of the velocity field alone is in fact an indispensable ingredient for the method. Indeed, energy inequalities are typically derived by testing the system against the solution itself. In the local case, one additionally multiplies the solution by a smooth cut-off function. But the solution times an arbitrary test function of course does not produce in general a solenoidal field. Hence, an integration by parts in this context would require global knowledge of the pressure field as it does not drop out as in the case of weak solutions. To bypass this problem is exactly the purpose of the local pressure decomposition.

All in all, these ideas will eventually lead us to the notion of a *local suitable weak solution* due to Wolf [31]. Related ideas and "precursors" already appeared in [29, 30]. A rigorous definition and further details will be provided later in the text as this is exactly the content of Section 2.

1.2 Scaling symmetry of Navier–Stokes flows

Let us first take a look at smooth solutions $u\colon \mathbb{R}^d \times [0,T) \to \mathbb{R}$ of the heat equation

$$u_t - \Delta u = 0,$$

i.e. we consider the linear part of the Navier–Stokes equations ignoring the pressure for the time being. More precisely, we want to study the scaling properties of u and related energy quantities (cf. for the discussion in this subsection the blog post of Tao [26]).

It is immediate to verify that for all scaling parameters $\lambda > 0$ the function

$$u_\lambda\colon \mathbb{R}^d \times [0, \lambda^{-2}T) \to \mathbb{R}, \quad (x,t) \mapsto u(\lambda x, \lambda^2 t)$$

is again a smooth solution of the heat equation. In other words, the heat equation is scale-invariant (or obeys a scaling symmetry) with respect to the dilation scaling

$$S_\lambda\colon \mathbb{R}^d \times \mathbb{R}_{>0} \to \mathbb{R}^d \times \mathbb{R}_{>0}, \quad (x,t) \mapsto (\lambda x, \lambda^2 t),$$

which is also called the parabolic scaling of space-time $\mathbb{R}^d \times \mathbb{R}_{>0}$. If one thinks of λ as being small, then this operation simply represents a kind of ideal microscope which enables us to zoom in on the "small scales" and to study the behaviour of the solution u at these scales via u_λ. As the heat equation is linear, we also obtain a scaling symmetry for every $\alpha \geq 0$ via

$$S_{\lambda,\alpha} \colon \mathbb{R}^d \times \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}^d \times \mathbb{R}_{>0} \times \mathbb{R}, \quad (x,t,u) \mapsto (\lambda x, \lambda^2 t, \lambda^\alpha u),$$

i.e. we also included a scaling factor for the dependent variable $u \in \mathbb{R}$.

Now, under appropriate conditions for the behaviour of the solution at infinity, a simple integration by parts shows that the *maximal kinetic energy*

$$\operatorname*{ess\,sup}_{t\in[0,T)} \frac{1}{2} \int_{\mathbb{R}^d} |u(x,t)|^2 \,\mathrm{d}x \tag{1.7}$$

and the *cumulative energy dissipation*

$$\frac{1}{2} \int_0^T \int_{\mathbb{R}^d} |\nabla u(x,t)|^2 \,\mathrm{d}x \tag{1.8}$$

are bounded by the *initial kinetic energy*

$$E = \frac{1}{2} \int_{\mathbb{R}^d} |u(x,0)|^2 \,\mathrm{d}x < \infty. \tag{1.9}$$

In other words, both terms are globally controlled by the initial kinetic energy. If we now rescale u to $u_{\lambda,\alpha}$ via the dilation scaling $S_{\lambda,\alpha}$, the change of variables formula implies that the maximal kinetic energy and the cumulative energy dissipation of $u_{\lambda,\alpha}$ are bounded by

$$\lambda^{2\alpha-d} E.$$

Hence, if $\alpha > d/2$ we observe that the control of the solution through the kinetic energy and the energy dissipation more and more improves as we move into the small scales. On the other side, if $\alpha < d/2$ both quantities lose their appeal for the study of the behaviour of u at small scales as they may be less and less able to control the solution. Finally, in the remaining regime $\alpha = d/2$ both quantities make no difference between small and large scales.

In the first case, we say that the maximal kinetic energy and the cumulative energy dissipation are *subcritical* quantities (with respect to the scaling $S_{\lambda,\alpha}$). In the second case, we call them instead *supercritical* quantities and in the last one *critical* quantities. Of course one may insist at this point that for the heat equation these considerations may not be too important due to the linearity of the problem. But this changes severely when one studies the Navier–Stokes equations.

In order to justify this, we consider in analogy to above the following scaling map

$$S_{\lambda,\alpha}\colon \mathbb{R}^3\times\mathbb{R}_{>0}\times\mathbb{R}^3\times\mathbb{R}\longrightarrow\mathbb{R}^3\times\mathbb{R}_{>0}\times\mathbb{R}^3\times\mathbb{R}$$
$$(x,t,\mathbf{u},p)\longmapsto(\lambda x,\lambda^2 t,\lambda^\alpha\mathbf{u},\lambda^{\alpha+1}p),$$

which also acts on the dependent variable $p\in\mathbb{R}$ and reflects the fact that the velocity field is vector-valued. A simple calculation shows that if we rescale a smooth solution $(\mathbf{u},p)$ of the 3D Navier–Stokes equations (1.1) with respect to the dilation scaling $S_{\lambda,\alpha}$, the resulting pair of functions $(\mathbf{u}_{\lambda,\alpha},p_{\lambda,\alpha})$ then obeys the system

$$\partial_t\mathbf{u}_{\lambda,\alpha}-\Delta\mathbf{u}_{\lambda,\alpha}+\lambda^{-\alpha+1}\operatorname{div}(\mathbf{u}_{\lambda,\alpha}\otimes\mathbf{u}_{\lambda,\alpha})=-\nabla p_{\lambda,\alpha}. \tag{1.10}$$

We observe that due to the presence of the non-linearity in the Navier–Stokes equations, we obtain a condition for the parameter $\alpha\geq 0$. As a matter of fact, unless one takes $\alpha=1$, the scaling map $S_{\lambda,\alpha}$ does not generate a scaling symmetry on the level of the PDE! This is in strong contrast to the situation above, where $\alpha\geq 0$ can be considered as a free parameter which itself is an artefact of the linearity of the heat equation. Just as bad, $\alpha=1$ corresponds in 3D to the supercritical regime for the maximal kinetic energy and the cumulative dissipation energy. On the other side, roughly speaking these are up to now the only known quantities which can be controlled globally in the three-dimensional case, which in turn is one of the main reasons why 1.000.000 \$ are still awaiting to be awarded. (Note also that in the two-dimensional situation $\alpha=1$ corresponds exactly to the critical regime for these quantities.)

As said above, in this work we study the local regularity (or more precisely, the local boundedness) of a certain subclass of weak solutions to the Navier–Stokes equations. As the Navier–Stokes equations are invariant with respect to the dilation scaling $S_{\lambda,1}$, it seems to be natural to consider those quantities in the formulation of regularity conditions which are invariant with respect to this scaling (cf. [21]).

Before we finally list some of these, let us introduce the following notation. For a point in space $x\in\mathbb{R}^3$ and $R>0$, we write $B_R(x)$ for the open ball with centre x and radius R, i.e. $B_R(x)=\{y\in\mathbb{R}^3\colon \|y-x\|_2<R\}$. Furthermore, we denote by $Q_R(x,t)$ the parabolic space-time cylinder with top-centre point $(x,t)\in\mathbb{R}^3\times\mathbb{R}$ and radius $R>0$, i.e.

$$Q_R(x,t)=B_R(x)\times(t-R^2,t).$$

If there is no danger for confusion, we occasionally suppress in the notation the dependence on the centre points and simply write B_R and Q_R, respectively.

With this notation at hand, examples of (local) scale-invariant (or in the terminology from above, critical) quantities in 3D are given by:

$$\frac{1}{R} \operatorname*{ess\,sup}_{s\in(t-R^2,t)} \int_{B_R(x)} |\mathbf{u}(y,s)|^2 \,\mathrm{d}y, \tag{1.11}$$

$$\frac{1}{R} \int_{Q_R(x,t)} |\nabla \mathbf{u}(y,s)|^2 \,\mathrm{d}y\mathrm{d}s, \tag{1.12}$$

$$\frac{1}{R^2} \int_{Q_R(x,t)} |\mathbf{u}(y,s)|^3 \,\mathrm{d}y\mathrm{d}s, \tag{1.13}$$

$$\frac{1}{R^3} \int_{Q_R(x,t)} |\mathbf{u}(y,s)|^2 \,\mathrm{d}y\mathrm{d}s. \tag{1.14}$$

In analogy to above, we will call the first one the *rescaled maximum kinetic energy*. It is also exactly the quantity we want to use in this work when formulating regularity criteria. The second quantity is called the *rescaled cumulative dissipation energy*.

1.3 Local regularity theory & main results

At the beginning, we want to review some parts of the theory of suitable weak solutions before we move on to state the main result of this work. We take as a starting point the partial regularity result due to Scheffer [18, 19], which loosely speaking states that there exists at least one weak solution such that the set of singular points has finite 5/3-dimensional Hausdorff measure. Here, a space-time point is called singular if the velocity field is not essentially bounded in any neighbourhood of that point.

In the celebrated paper of Caffarelli et al. [1], this result was significantly improved. In fact, it is shown that for all suitable weak solutions the one dimensional (parabolic) Hausdorff measure of the set of singular points vanishes. This still provides the best bound for the Hausdorff dimension of the set of singular points.

Such partial regularity results are proved by means of interior local regularity results for the Navier–Stokes equations. These results can be categorized into two types. (The following presentation closely resembles the discussion in [20].) The first type of statement contains an explicit smallness assumption on the pressure, whereas the second type appears to be independent of the pressure at first glance. Of course, this is wrong, but let us delay the discussion of this fact for the time being.

Certain is that for both types scale-invariant quantities take on the key role. The examples given at the end of the last subsection do not incorporate the pressure. But as the pressure is globally defined for a suitable weak solution, there is no reason why one should ignore it in general. For instance, in view of the definition of a suitable weak solution it is natural to consider the quantity

$$F(R) = \frac{1}{R^2} \int\limits_{Q_R(0,0)} |\mathbf{u}|^3 + |p|^{\frac{3}{2}} \,\mathrm{d}x\mathrm{d}t.$$

Then, the following result holds which is a prototype for the first kind of local regularity statements:

Let $(\mathbf{u}, p)$ *be a suitable weak solution of the Navier–Stokes equations in the unit parabolic cylinder* $Q = Q_1$. *There exists absolute constants* $\varepsilon > 0$ *and* $C > 0$ *such that whenever* $F(1) < \varepsilon$, *then* $\mathbf{u}$ *is bounded on the closure of* $Q_{1/2}$ *by* C. *Furthermore,* $\mathbf{u}$ *is Hölder continuous on the closure of* $Q_{1/2}$ *with respect to the parabolic metric.*

For a proof of this, we refer to [1, 13]. A corresponding statement including spatial derivatives of arbitrary degree can be found in [17].

On the other side, a prototype for the second kind of statements is formulated in terms of the cumulative (local) dissipation energy

$$B(R) = \frac{1}{R} \int\limits_{Q_R(0,0)} |\nabla \mathbf{u}|^2 \,\mathrm{d}x\mathrm{d}t.$$

The first part of this result also already appeared in [1]. For an alternative approach, one can consult Lin [15].

Let $(\mathbf{u}, p)$ *be a suitable weak solution of the Navier–Stokes equations in the unit parabolic cylinder* $Q = Q_1$. *Then, there exist an absolute constant* $\varepsilon > 0$ *such that if* $\limsup_{R\to 0} B(R) < \varepsilon$, *then* $z = (0,0)$ *is a regular point of* $\mathbf{u}$. *Furthermore, there is a radius* $0 < R < 1$ *such that the velocity field is Hölder continuous on the closure of* Q_R *with respect to the parabolic metric.*

As mentioned above, one may think that this theorem makes no assumption on the pressure. But in this kind of local regularity statements, this assumption is hidden in the definition of a suitable weak solution, i.e. $p \in L^{3/2}(Q)$ and the magnitude of the radius R actually hinges on this global information.

Another class of scale-invariant quantities which admit local regularity results of the second type is given by the so-called Ladyzhenskaya–Prodi–Serrin quantities

$$\frac{1}{R^{\frac{3}{s}+\frac{2}{l}-1}} \left(\int_{-R^2}^{0} \left(\int_{B_R(0)} |\mathbf{u}|^s \,\mathrm{d}x \right)^{\frac{l}{s}} \mathrm{d}t \right)^{\frac{1}{l}}, \quad \frac{3}{s} + \frac{2}{l} \leq 1, \quad s \geq 3.$$

There is an extensive literature on these quantities, e.g. [4, 12, 22, 24, 25]. Similar quantities involving the vorticity $\omega = \operatorname{curl} \mathbf{u}$ are treated in [10, 29].

Now, let us finally state the main result of this work which is an interior local regularity result for local suitable weak solutions. If one insists on classifying our statement into one of the two categories discussed above, then it would be the second one. But in our case, there really is no assumption on the pressure, just because in our situation there is simply no hope for a globally defined one. This has a price! It is a common feature of interior local regularity results that in the absence of any assumptions on the pressure, one may loose continuity for the time direction. Therefore, one should label our theorem more precisely as a local boundedness result.

Theorem 1.4. *Let $\mathbf{u}$ be a local suitable weak solution of the Navier–Stokes equations* (1.1) *in the space-time cylinder $Q = \Omega \times (a, b)$. Then, there exist absolute constants ε, K_0 and K_1 with the property that for all parabolic cylinders $Q_R = Q_R(x_0, t_0) \subset Q$*

$$\frac{1}{R}\|\mathbf{u}\|^2_{L^{2,\infty}(Q_R)} \leq \varepsilon^2 \quad \Longrightarrow \quad \begin{cases} \|\mathbf{u}\|^2_{L^\infty(Q_{R/2})} & \leq \frac{K_0}{R^2}\left(\frac{1}{R}\|\mathbf{u}\|^2_{L^{2,\infty}(Q_R)}\right) \\ \|\nabla\mathbf{u}\|^2_{L^\infty(Q_{R/2})} & \leq \frac{K_1}{R^4}\left(\frac{1}{R}\|\mathbf{u}\|^2_{L^{2,\infty}(Q_R)}\right). \end{cases} \tag{1.15}$$

Furthermore, the velocity field is Hölder continuous in the spatial variables on the closure of $B_{R/2}(x_0)$.

We want to emphasize that this result is *not* contained in any of the works listed above, e.g. [10]. Again, this is simply due to the fact that we do not incorporate any global control on the pressure which is present in all of the works mentioned above.

Proof. The asserted bounds are essentially a combination of Theorem 4.1 and Theorem 5.1. The continuity statement follows immediately from the discussion of Example 5.4. ∎

For more on local ε-regularity theory for local suitable weak solutions, we refer to Wolf [31].

1.4 Structure of the thesis

The present work is organized as follows. In Section 2, we begin with the notion of a local suitable weak solution. To this end, the already mentioned local pressure decomposition will be carried out based on a result for the steady Stokes system. In order to motivate the generalized local energy inequality occurring in the definition of a local suitable weak solution, we prefer to give at first some purely formal computations in order not to clutter the presentation through technical details. Afterwards, we will comment briefly on how to make certain steps rigorous in these formal considerations.

Furthermore, we discuss the scaling properties of local suitable weak solutions and the regularity properties of the pressure terms. We also include a warning example which shall remind us that in general we cannot expect more regularity in the time direction than boundedness for a local suitable weak solution.

Section 4 contains the heart of the presentation, i.e. a local L^∞-bound of the velocity field in terms of the *rescaled maximum kinetic energy*. For this bound to hold, we have to assume that the rescaled maximum kinetic energy itself is sufficiently small. In other words, our bound represents a local ε-regularity result. For the proof of our local L^∞-bound, we invoke an inductive argument in the spirit of [1]. The induction basis essentially follows from a Caccioppoli inequality which is tailored to this purpose. This Caccioppoli inequality is the subject of Section 3. For the induction step, we basically use the generalized local energy inequality in conjunction with a suitable choice of test function. With view on (1.6), it is somehow natural to think about the fundamental solution of the backward heat equation in this context. More explanations and motivation will be provided in Section 4.

In Section 5, we then prove the local L^∞-bound for the spatial partial derivatives of the velocity field. The proof can be based on a general local boundedness result for the Navier–Stokes type system

$$\partial_t \mathbf{u} - \Delta \mathbf{u} + \operatorname{div}(\mathbf{u} \otimes \mathbf{A}) = -\nabla p, \quad \operatorname{div} \mathbf{u} = 0,$$

where $\mathbf{A}$ is a bounded field. Therefore, we will first concentrate on this system and then discuss the application. All proofs given in Section 5 are somehow similar in style to what we already discussed before.

Finally, we collect in an appendix for reference purposes and better readability some important techniques, results and constructions which occur on different occasions throughout the text.

2 Local suitable weak solutions

In this section, we want to introduce and motivate the notion of a *local suitable weak solution* of the Navier–Stokes equations. This notion was first introduced by Wolf in [31] and requires some known facts about the steady Stokes system. In this respect, it is fair to say that the following result on the steady Stokes system is the linchpin around which everything eventually revolves in this work. For a proof of this result, see [7].

Proposition 2.1. *Let $U \subset \mathbb{R}^3$ be a bounded domain with C^1 boundary ∂U and $1 < q < \infty$. For every $\mathbf{F} \in W^{-1,q}(U,\mathbb{R}^3)$ there exist uniquely determined functions $\mathbf{u} \in W^{1,q}_{0,\sigma}(U,\mathbb{R}^3)$ and $p \in L^q_0(U)$ such that*

$$-\Delta \mathbf{u} + \nabla p = \mathbf{F} \tag{2.1}$$

holds true as an equation in $W^{-1,q}(U,\mathbb{R}^3)$. Furthermore, there exists a constant $c = c(q,U) > 0$ such that

$$\|\nabla \mathbf{u}\|_{L^q(U)} + \|p\|_{L^q(U)} \le c\|\mathbf{F}\|_{W^{-1,q}(U)}. \tag{2.2}$$

In particular,

$$W^{-1,q}(U,\mathbb{R}^3) = W^{-1,q}_{\mathrm{div}}(U,\mathbb{R}^3) \oplus W^{-1,q}_{\mathrm{grad}}(U,\mathbb{R}^3),$$

with $W^{-1,q}_{\mathrm{div}}(U,\mathbb{R}^3) = \{-\Delta \mathbf{u} \colon \mathbf{u} \in W^{1,q}_{0,\sigma}(U,\mathbb{R}^3)\}$ and $W^{-1,q}_{\mathrm{grad}}(U,\mathbb{R}^3) = \{\nabla p \colon p \in L^q_0(U)\}$.

Definition 2.2. Let $U \subset \mathbb{R}^3$ be a bounded domain with C^1 boundary ∂U and fix $1 < q < \infty$. We denote by $P_{q,U}$ the bounded linear operator

$$P_{q,U} \colon W^{-1,q}(U,\mathbb{R}^3) \longrightarrow L^q_0(U), \quad \mathbf{F} \longmapsto p, \tag{2.3}$$

where p is the uniquely determined function in $L^q_0(U)$ with the property that $\mathbf{F} - \nabla p \in W^{-1,q}_{\mathrm{div}}(U,\mathbb{R}^3)$. Obviously, the operator $P_{q,U}$ is onto and we are equipped with the bound $\|P_{q,U}\| \le c(q,U)$. Furthermore, let us denote by $R_{q,U}$ the projection of $W^{-1,q}(U,\mathbb{R}^3)$ onto $W^{-1,q}_{\mathrm{grad}}(U,\mathbb{R}^3)$, i.e.

$$R_{q,U} \colon W^{-1,q}(U,\mathbb{R}^3) \longrightarrow W^{-1,q}_{\mathrm{grad}}(U,\mathbb{R}^3), \quad \mathbf{F} \longmapsto \nabla P_{q,U}\mathbf{F}. \tag{2.4}$$

◊

In a next step, we want to extend the definition of the operator $P_{q,U}$ to functions in $L^s(a,b;W^{-1,q}(U,\mathbb{R}^3))$ with $1 \le s \le \infty$ (and $1 < q < \infty$). To this end, recall first that the space $W^{1,q}_0(U,\mathbb{R}^3)$ is a separable and reflexive Banach space. Furthermore,

$$W^{-1,q}(U,\mathbb{R}^3)' \cong W^{1,q}_0(U,\mathbb{R}^3)$$

and therefore $W^{-1,q}(U,\mathbb{R}^3)$ is a separable Banach space too, since any Banach space with separable dual space is separable. Due to Pettis, this means that the notions of strong measurability and weak measurability of a map

$$\mathbf{F}\colon (a,b) \to W^{-1,q}(U,\mathbb{R}^3)$$

actually coincide. The same can be concluded for functions with values in $L^q_0(U)$ or $W^{-1,q}_{\mathrm{grad}}(U,\mathbb{R}^3)$. Consequently, the following definitions in (2.5) and (2.6) are meaningful.

Definition 2.3. Let $U \subset \mathbb{R}^3$ be a bounded domain with C^1 boundary ∂U and fix $1 < q < \infty$ as well as $1 \le s \le \infty$. We denote by

$$\mathscr{P}_{q,U}\colon L^s\big(a,b;W^{-1,q}(U,\mathbb{R}^3)\big) \longrightarrow L^s\big(a,b;L^q_0(U)\big)$$

the bounded and surjective linear operator defined via

$$(\mathscr{P}_{q,U}\mathbf{F})(t) := P_{q,U}\mathbf{F}(t) \quad \text{for a.e. } t \in (a,b). \tag{2.5}$$

Analogously, we write

$$\mathscr{R}_{q,U}\colon L^s\big(a,b;W^{-1,q}(U,\mathbb{R}^3)\big) \longrightarrow L^s\big(a,b;W^{-1,q}_{\mathrm{grad}}(U,\mathbb{R}^3)\big)$$

for the bounded and surjective linear operator given by

$$(\mathscr{R}_{q,U}\mathbf{F})(t) := R_{q,U}\mathbf{F}(t) \quad \text{for a.e. } t \in (a,b). \tag{2.6}$$

Furthermore, if $\mathbf{F} \in L^s\big(a,b;W^{-1,q}(U,\mathbb{R}^3)\big)$ we define

$$\mathscr{P}_{q,U}\partial_t\mathbf{F} := \partial_t\mathscr{P}_{q,U}\mathbf{F}, \quad \mathscr{R}_{q,U}\partial_t\mathbf{F} := \partial_t\mathscr{R}_{q,U}\mathbf{F}, \tag{2.7}$$

where the right hand sides are understood in the distributional sense. ◇

Now, consider a weak solution $\mathbf{u}$ to the Navier–Stokes equations. Recall that this in particular entails finite energy, i.e.

$$\mathbf{u} \in L^2\big(a,b;W^{1,2}(\Omega,\mathbb{R}^3)\big) \cap L^\infty\big(a,b;L^2(\Omega,\mathbb{R}^3)\big).$$

To motivate the definition of a local suitable weak solution, we proceed with some *purely formal* calculations. To this end, fix a point $x_0 \in \Omega$ and $0 < R < \mathrm{dist}(x_0,\partial\Omega)$, i.e. $B := B_R(x_0) \subset\subset \Omega$. Applying the projector $\mathscr{R}_{3/2,B}$ to the Navier–Stokes equations yields

$$\partial_t\mathscr{R}_{3/2,B}\mathbf{u} - \mathscr{R}_{3/2,B}\Delta\mathbf{u} + \mathscr{R}_{3/2,B}\mathrm{div}(\mathbf{u}\otimes\mathbf{u}) = -\mathscr{R}_{3/2,B}\nabla p = -\nabla p. \tag{2.8}$$

(At least formally, one may think of the pressure field as being normalized such that it integrates to zero over B.) Now, introducing

$$p_{h,B} := -\mathscr{P}_{3/2,B}\mathbf{u} = -\mathscr{P}_{2,B}\mathbf{u}, \tag{2.9}$$

$$p_{1,B} := -\mathscr{P}_{3/2,B}\mathrm{div}(\mathbf{u}\otimes\mathbf{u}), \tag{2.10}$$

$$p_{2,B} := \mathscr{P}_{3/2,B}\Delta\mathbf{u} = \mathscr{P}_{2,B}\Delta\mathbf{u} \tag{2.11}$$

and applying the operator $\mathscr{P}_{3/2,B}$ to equation (2.8), we obtain a type of local pressure decomposition in terms of the velocity field $\mathbf{u}$ alone, or more precisely

$$p = \partial_t p_{h,B} + p_{0,B}, \quad p_{0,B} := p_{2,B} - p_{1,B}. \tag{2.12}$$

In addition, equation (2.8) also implies

$$\partial_t \mathbf{v}_B - \Delta \mathbf{u} + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) = -\nabla p_{0,B}, \tag{2.13}$$

where the function $\mathbf{v}_B$ is given by

$$\mathbf{v}_B := \mathbf{u} + \nabla p_h. \tag{2.14}$$

Now, consider a test function $\phi \in C_c^\infty(B \times (a,b))$ and test the system (2.13) against $\boldsymbol{\varphi} = 2\phi \mathbf{v}_B$. Integration by parts then yields for a.e. $t \in (a,b)$ the following generalized local energy equality encoded in terms of the velocity field

$$\begin{aligned} &\int_B |\mathbf{v}_B(t)|^2 \phi(t)\, \mathrm{d}x + 2\int_a^t \int_B |\nabla \mathbf{v}_B|^2 \phi \, \mathrm{d}x\mathrm{d}s = \\ &= \int_a^t \int_B |\mathbf{v}_B|^2 (\partial_t \phi + \Delta \phi)\, \mathrm{d}x\mathrm{d}s + \int_a^t \int_B \big(|\mathbf{u}|^2 + 2p_{0,B}\big)\mathbf{v}_B \cdot \nabla\phi \, \mathrm{d}x\mathrm{d}s + \\ &\quad + 2\int_a^t \int_B (\mathbf{u} \otimes \mathbf{u}) : \nabla(\phi \nabla p_{h,B})\, \mathrm{d}x\mathrm{d}s - \int_a^t \int_B |\mathbf{u}|^2 \nabla p_{h,B} \cdot \nabla \phi \, \mathrm{d}x\mathrm{d}s. \end{aligned}$$

We will now turn this property into a postulate, replacing the equality with an inequality. The existence of local suitable weak solutions was proven in [31, Theorem 3.2].

Definition 2.4. Let $\mathbf{u}$ be a weak solution to the Navier–Stokes equations in the space-time cylinder $\Omega \times (a,b)$. We call $\mathbf{u}$ a local suitable weak solution, if for every ball $B \subset\subset \Omega$, every non-negative $\phi \in C_c^\infty(B \times (a,b))$ and for a.e. $t \in (a,b)$ the following inequality holds true

$$\begin{aligned} &\int_B |\mathbf{v}_B(t)|^2 \phi(t)\, \mathrm{d}x + 2\int_a^t \int_B |\nabla \mathbf{v}_B|^2 \phi \, \mathrm{d}x\mathrm{d}s \le \\ &\le \int_a^t \int_B |\mathbf{v}_B|^2 (\partial_t \phi + \Delta \phi)\, \mathrm{d}x\mathrm{d}s + \int_a^t \int_B \big(|\mathbf{u}|^2 + 2p_{0,B}\big)\mathbf{v}_B \cdot \nabla\phi \, \mathrm{d}x\mathrm{d}s + \\ &\quad + 2\int_a^t \int_B (\mathbf{u} \otimes \mathbf{u}) : \nabla(\phi \nabla p_{h,B})\, \mathrm{d}x\mathrm{d}s - \int_a^t \int_B |\mathbf{u}|^2 \nabla p_{h,B} \cdot \nabla \phi \, \mathrm{d}x\mathrm{d}s. \end{aligned} \tag{2.15}$$

We will refer to this inequality in the following as the generalized local energy inequality. ♢

We conclude this section with a series of remarks (including one warning example) concerning the notion of a local suitable weak solution.

Remark 2.5. The operators $P_{q,\Omega}$ satisfy certain regularity properties which can be summarized as follows (cf. [20]).

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, $k \in \mathbb{N}_0$, $1 < q < \infty$ and $\mathbf{F} \in W^{k,q}(\Omega)$. Assume that functions $\mathbf{u} \in W^{1,q}_{0,\sigma}(\Omega)$ and $p \in L^q(\Omega)$ satisfy the non-homogeneous Stokes problem in Ω*

$$-\Delta \mathbf{u} + \nabla p = \mathbf{F}, \quad \operatorname{div} \mathbf{u} = 0.$$

Then, in fact we have $\nabla^2 \mathbf{u} \in W^{k,q}(\Omega)$ and $\nabla p \in W^{k,q}(\Omega)$. Also, the following estimate holds

$$\|\nabla^2 \mathbf{u}\|^2_{W^{k,q}(\Omega)} + \|\nabla p\|^2_{W^{k,q}(\Omega)} \lesssim \|\mathbf{F}\|^2_{W^{k,q}(\Omega)}. \tag{2.16}$$

The proportionality constant implicit in this bound depends on k, q and the geometry of Ω only.

In the following, we will refer to the bounds in (2.16) and (2.2) as the *Cattabriga–Solonnikov estimates* (cf. [20]). ♢

Remark 2.7. Together with (a slightly modified version of) interpolation inequality (A.10), the Cattabriga–Solonnikov estimates guarantee that all of the integrals on the right-hand side of the generalized local energy inequality are finite. ♢

Remark 2.8. In this remark, we want to investigate the scaling behaviour of local suitable weak solutions of the Navier–Stokes equations. To this end, let us first introduce some notation. As in the introduction, we denote by S_λ the scaling map associated to the parabolic scaling of space-time $\mathbb{R}^3 \times \mathbb{R}$, i.e.

$$S_\lambda \colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}, \quad (x_1, x_2, x_3, t) \mapsto (\lambda x_1, \lambda x_2, \lambda x_3, \lambda^2 t).$$

For a vector field $\mathbf{f} \colon Q \to \mathbb{R}^m$ defined on a space-time cylinder $Q = \Omega \times (a, b)$, we then define $Q_\lambda = \lambda^{-1}\Omega \times (\lambda^{-2} a, \lambda^{-2} b)$ and

$$S_\lambda \mathbf{f} \colon Q_\lambda \to \mathbb{R}^m, (x_1, x_2, x_3, t) \mapsto \mathbf{f}\big(S_\lambda(x_1, x_2, x_3, t)\big).$$

(This scaling operation is lifted to distributions in the obvious way by duality.) Finally, we introduce the shorthand

$$\mathbf{f}_\lambda := \lambda S_\lambda \mathbf{f}.$$

With this notation at our disposal, it is straightforward to show from the definitions that for every local suitable weak solution $\mathbf{u}$ of the Navier–Stokes equations the following scaling relations hold (for all $B \subset\subset \Omega$):

$$-\mathscr{P}_{2,\lambda^{-1}B}\mathbf{u}_\lambda = S_\lambda p_{h,B}, \quad -\mathscr{P}_{3/2,\lambda^{-1}B}\operatorname{div}\big(\mathbf{u}_\lambda \otimes \mathbf{u}_\lambda\big) = \lambda^2 S_\lambda p_{1,B}$$
$$\mathscr{P}_{2,\lambda^{-1}B}\Delta\mathbf{u}_\lambda = \lambda^2 S_\lambda p_{2,B}.$$

In particular, we obtain

$$-\nabla\mathscr{P}_{2,\lambda^{-1}B}\mathbf{u}_\lambda = (\nabla p_{h,B})_\lambda.$$

Thus, a simple dimension count in the generalized local energy inequality (2.15) verifies that these scaling relations for the local pressure terms correspond exactly to what is needed in order to infer the validity of (2.15) for $\mathbf{u}_\lambda$ from the validity of (2.15) for $\mathbf{u}$.

All in all, this proves that a vector field $\mathbf{u}\colon Q \to \mathbb{R}^3$ is a local suitable weak solution of the Navier–Stokes equations (in Q) if and only if $\mathbf{u}_\lambda$ has this property (in Q_λ). Note also that the scaling behaviour of the local pressure terms and of the function $\mathbf{v}_B$ corresponds exactly to what one would expect in view of equation (2.13). ♢

⚠ *Example 2.9.* The notion of a local suitable weak solution does not require any regularity assumptions on the boundary of $\Omega \subset \mathbb{R}^3$, as we already noted in the introduction. This is also directly related to the fact that there is no need to define the pressure field *globally*. Instead, we just work with terms which are *locally* defined through the velocity field in the interior of the domain Ω. This generality has its price, as the following example shows us plainly!

Let $\mathbf{b} \in L^\infty\big((-1,0),\mathbb{R}^3\big)$ be *any* essentially bounded vector-valued function and put

$$\mathbf{u}(x,t) := \mathbf{b}(t), \quad (x,t) \in B_1(0) \times (-1,0).$$

It is straightforward to show that $\mathbf{u}$ is a weak solution of the Navier–Stokes equations in the unit parabolic cylinder. Furthermore, we compute

$$p_{1,B} = p_{2,B} \equiv 0, \quad p_{h,B}(x,t) = -\mathbf{b}(t)\cdot x.$$

(Of course, the latter term has to be normalized such that $p_{h,B}$ really integrates to zero over B. But this is of no importance for us here.) In particular, we obtain

$$\mathbf{v}_B = \mathbf{u} + \nabla p_{h,B} \equiv 0.$$

Thus, $\mathbf{u}$ is indeed a local suitable weak solution of the Navier–Stokes equations in the unit parabolic cylinder. This example therefore shows that in the time direction we cannot expect any more regularity than boundedness for a local suitable weak solution. ♢

Remark 2.10. We derived the system (2.13) by purely formal computations. But the motivating argument for the definition of a local suitable weak solution can be made rigorous. For a precise statement and a proof, we refer to [31, Lemma 2.4].

Let us mention at least that one serious technical issue one has to deal with stems from the fact that (as we already discussed in the example preceding this remark) the function $\mathbf{v}_B$ may not have enough regularity in time in order to justify any sort of integration by parts involving the time derivative. Thus, one has to regularize in time, and one prominent option to do so is the Steklov average (cf. the proof of Lemma 2.4 in [31]). $\diamondsuit$

3 A Caccioppoli inequality

In the following, we establish a Caccioppoli inequality for local suitable weak solutions of the Navier–Stokes equations which is formulated in terms of the following scale-invariant quantities

(3.1) $$B(R) = R^{-1}\|\nabla \mathbf{u}\|^2_{L^2(Q_R)}, \quad C(R) = R^{-1}\|\mathbf{u}\|^2_{L^{2,\infty}(Q_R)},$$

(3.2) $$E(R) = R^{-\frac{4}{3}}\|\mathbf{u}\|^2_{L^3(Q_R)}.$$

Lemma 3.1. *Let* $\mathbf{u}$ *be a local suitable weak solution to* (1.1) *in the space-time cylinder* $Q = \Omega \times (a, b)$. *Furthermore, consider* $R > 0$ *and* $(x_0, t_0) \in Q$ *such that* $Q_R = B_R(x_0) \times (t_0 - R^2, t_0) \subset Q$. *Then, the following Caccioppoli inequality is satisfied*

(3.3) $$B(R/2) + C(R/2) + E(R/2) \lesssim C(R)^3 + C(R).$$

The proportionality constant implicit in this bound is an absolute constant.

Proof. We follow the argumentation in [31] and make appropriate adjustments where needed. We choose arbitrary real numbers $R/2 \le \rho < r \le R$ and put $\sigma = (r+\rho)/2$. Now, let $\phi \in C_c^\infty(Q)$ be a smooth and in Q compactly supported function with the following properties:

i) $0 \le \phi \le 1$ in Q,

ii) $\phi \equiv 0$ in $Q \setminus B_\sigma \times (t_0 - \sigma^2, t_0 + \sigma^2)$,

iii) $\phi \equiv 1$ in Q_ρ and

iv) $|\phi_t| + |\Delta\phi| + |\nabla\phi|^2 \lesssim (r-\rho)^{-2}$ uniformly in Q.

Finally, put $\tilde{\sigma} = (\sigma + r)/2$. Let us also abbreviate for what follows $\mathbf{v} = \mathbf{v}_{B_{\tilde{\sigma}}}$ and so on. In other words, we carry out the local pressure decomposition on the ball $B_{\tilde{\sigma}}$.

Note that we may insert the test function ϕ^2 into the generalized local energy inequality (2.15) in the definition of a local suitable weak solution. In particular, we obtain the following bound

(3.4) $$\|\phi\mathbf{v}\|^2_{L^{2,\infty}(Q_\sigma)} + \|\phi\nabla\mathbf{v}\|^2_{L^2(Q_\sigma)} \lesssim J_1 + J_2 + J_3 + J_4,$$

where we introduced the following quantities

$$J_1 = \int_{Q_\sigma} |\mathbf{v}|^2|\phi_t + \Delta\phi| \,\mathrm{d}x\mathrm{d}t, \quad J_2 = \int_{Q_\sigma} \phi|\mathbf{u}|^2|\mathbf{v}||\nabla\phi| \,\mathrm{d}x\mathrm{d}t,$$

$$J_3 = \int_{Q_\sigma} |p_0||\mathbf{v}||\nabla\phi| \,\mathrm{d}x\mathrm{d}t, \quad J_4 = \int_{Q_\sigma} |\mathbf{u}|^2|\nabla(\phi\nabla p_h)| + \phi|\mathbf{u}|^2|\nabla p_h||\nabla\phi| \,\mathrm{d}x\mathrm{d}t.$$

After these preparations, we will derive appropriate bounds for each of these terms. We begin with J_1 and obtain the bound

$$J_1 \lesssim (r-\rho)^{-2} \|\mathbf{u}\|^2_{L^2(Q_r)}, \tag{3.5}$$

which is an immediate consequence of the Cattabriga–Solonnikov estimates for the steady Stokes system.

Along the same lines, we derive the following bound for the second term making use of Hölder's inequality

$$J_2 \lesssim (r-\rho)^{-1} \|\mathbf{u}\|^3_{L^3(Q_r)}.$$

In order to get rid of the L^3-norm, we exploit the interpolation inequality (A.10) given in the appendix as follows

$$\begin{aligned}(r-\rho)^{-1}\|\mathbf{u}\|^3_{L^3(Q_r)} &\lesssim \frac{r^{\frac{1}{2}}}{r-\rho}\|\mathbf{u}\|^{\frac{3}{2}}_{L^{2,\infty}(Q_r)}\left(\|\mathbf{u}\|^2_{L^{2,\infty}(Q_r)} + \|\nabla\mathbf{u}\|^2_{L^2(Q_r)}\right)^{\frac{3}{4}} \\ &\lesssim \varepsilon^{-1}\frac{r^2}{(r-\rho)^4}\|\mathbf{u}\|^6_{L^{2,\infty}(Q_r)} + \varepsilon^{\frac{1}{3}}\left\{\|\mathbf{u}\|^2_{L^{2,\infty}(Q_r)} + \|\nabla\mathbf{u}\|^2_{L^2(Q_r)}\right\}.\end{aligned} \tag{3.6}$$

Here, we also invoked Young's inequality and $0 < \varepsilon \leq 1$ denotes, up to the moment, an arbitrary real number. All in all, this yields the bound

$$J_2 \lesssim \varepsilon^{-1}\frac{r^2}{(r-\rho)^4}\|\mathbf{u}\|^6_{L^{2,\infty}(Q_r)} + \varepsilon^{\frac{1}{3}}\left\{\|\mathbf{u}\|^2_{L^{2,\infty}(Q_r)} + \|\nabla\mathbf{u}\|^2_{L^2(Q_r)}\right\}. \tag{3.7}$$

For the term J_4, we start with the bound

$$J_4 \lesssim (r-\rho)^{-1}\|\mathbf{u}\|^2_{L^3(Q_\sigma)}\|\nabla p_h\|_{L^3(Q_\sigma)} + \|\mathbf{u}\|^2_{L^3(Q_\sigma)}\|\nabla^2 p_h\|_{L^3(Q_\sigma)}.$$

The term involving the second derivative of p_h can be treated with by means of the generalized L^p-Caccioppoli inequality (A.9). More precisely, the following bound is satisfied

$$\|\nabla^2 p_h\|_{L^3(Q_\sigma)} \lesssim \frac{\tilde{\sigma}^{\frac{3}{2}}}{(\tilde{\sigma}-\sigma)^{\frac{5}{2}}}\|\nabla p_h\|_{L^3(Q_{\tilde{\sigma}})} \lesssim \frac{r^{\frac{3}{2}}}{(r-\rho)^{\frac{5}{2}}}\|\mathbf{u}\|_{L^3(Q_r)}.$$

Apart from that, it suffices to make use of the Cattabriga–Solonnikov estimates and the interpolation argument in (3.6) in order to obtain

$$\begin{aligned}J_4 \lesssim \varepsilon^{-1}&\left(\frac{r^2}{(r-\rho)^4} + \frac{r^8}{(r-\rho)^{10}}\right)\|\mathbf{u}\|^6_{L^{2,\infty}(Q_r)} + \\ &+ \varepsilon^{\frac{1}{3}}\left\{\|\mathbf{u}\|^2_{L^{2,\infty}(Q_r)} + \|\nabla\mathbf{u}\|^2_{L^2(Q_r)}\right\}.\end{aligned} \tag{3.8}$$

It remains to bound the pressure terms, which is in this case just a matter of the corresponding Cattabriga–Solonnikov estimates. Hence,

$$\int_{Q_\sigma} |p_1||\mathbf{v}||\nabla\phi| \,\mathrm{d}x\mathrm{d}t \lesssim (r-\rho)^{-1}\|\mathbf{u}\|^3_{L^3(Q_r)}$$

and also

$$\int_{Q_\sigma} |p_2||\mathbf{v}||\nabla\phi| \,\mathrm{d}x\mathrm{d}t \lesssim (r-\rho)^{-1}\|\mathbf{u}\|_{L^2(Q_r)}\|\nabla\mathbf{u}\|_{L^2(Q_r)}$$
$$\lesssim \varepsilon^{-1}\frac{r^2}{(r-\rho)^2}\|\mathbf{u}\|^2_{L^{2,\infty}(Q_r)} + \varepsilon\|\nabla\mathbf{u}\|^2_{L^2(Q_r)}.$$

Together, these two bounds imply

$$J_3 \lesssim \varepsilon^{-1}\left\{\frac{r^2}{(r-\rho)^2}\|\mathbf{u}\|^2_{L^{2,\infty}(Q_r)} + \frac{r^2}{(r-\rho)^4}\|\mathbf{u}\|^6_{L^{2,\infty}(Q_r)}\right\} + + \varepsilon^{\frac{1}{3}}\left\{\|\mathbf{u}\|^2_{L^{2,\infty}(Q_r)} + \|\nabla\mathbf{u}\|^2_{L^2(Q_r)}\right\}. \tag{3.9}$$

At this point, we want to summarize what we have achieved so far. In view of (3.4), we can infer from our bounds from above that

$$\begin{aligned}&\|\phi\mathbf{v}\|^2_{L^{2,\infty}(Q_\sigma)} + \|\phi\nabla\mathbf{v}\|^2_{L^2(Q_\sigma)}\\ &\lesssim \varepsilon^{-1}\left\{\frac{r^2}{(r-\rho)^2}\|\mathbf{u}\|^2_{L^{2,\infty}(Q_r)} + \left(\frac{r^2}{(r-\rho)^4}+\frac{r^8}{(r-\rho)^{10}}\right)\|\mathbf{u}\|^6_{L^{2,\infty}(Q_r)}\right\} +\\ &\quad + \varepsilon^{\frac{1}{3}}\left\{\|\mathbf{u}\|^2_{L^{2,\infty}(Q_r)} + \|\nabla\mathbf{u}\|^2_{L^2(Q_r)}\right\}.\end{aligned} \tag{3.10}$$

In the last step, roughly speaking we want to replace $\mathbf{v}$ with $\mathbf{u}$ in this inequality. In order to do this, first note that due to the classical Caccioppoli inequality for harmonic functions

$$\|\nabla\mathbf{u}\|^2_{L^2(Q_\rho)} \lesssim \|\phi\nabla\mathbf{v}\|^2_{L^2(Q_\rho)} + \|\nabla^2 p_h\|^2_{L^2(Q_\rho)}$$
$$\lesssim \|\phi\nabla\mathbf{v}\|^2_{L^2(Q_\rho)} + (r-\rho)^{-2}\|\mathbf{u}\|^2_{L^2(Q_r)}.$$

Furthermore, applying again the interpolation inequality (A.10), we also obtain

$$\frac{1}{(R/2)^{\frac{1}{3}}}\|\mathbf{u}\|^2_{L^3(Q_\rho)} \lesssim \left(\frac{\rho}{R}\right)^{\frac{1}{3}}\|\mathbf{u}\|_{L^{2,\infty}(Q_\rho)}\left(\|\mathbf{u}\|^2_{L^{2,\infty}(Q_\rho)} + \|\nabla\mathbf{u}\|^2_{L^2(Q_\rho)}\right)^{\frac{1}{2}}$$
$$\lesssim \|\mathbf{u}\|^2_{L^{2,\infty}(Q_\rho)} + \|\nabla\mathbf{u}\|^2_{L^2(Q_\rho)}.$$

Choosing $\varepsilon > 0$ small enough in (3.10) consequently yields the bound

$$\begin{aligned}&\|\nabla\mathbf{u}\|^2_{L^2(Q_\rho)} + \frac{1}{(R/2)^{\frac{1}{3}}}\|\mathbf{u}\|^2_{L^3(Q_\rho)} \lesssim\\ &\lesssim \left(1+\frac{R^2}{(r-\rho)^2}\right)\|\mathbf{u}\|^2_{L^{2,\infty}(Q_R)} + \left(\frac{R^2}{(r-\rho)^4}+\frac{R^8}{(r-\rho)^{10}}\right)\|\mathbf{u}\|^6_{L^{2,\infty}(Q_R)} +\\ &\quad + \frac{1}{2}\left\{\|\nabla\mathbf{u}\|^2_{L^2(Q_r)} + \frac{1}{(R/2)^{\frac{1}{3}}}\|\mathbf{u}\|^2_{L^3(Q_r)}\right\}.\end{aligned}$$

Hence, we are in position to exploit the iteration argument from Proposition A.1 in the appendix which entails the bound

$$\|\nabla \mathbf{u}\|^2_{L^2(Q_{R/2})} + \frac{1}{(R/2)^{\frac{1}{3}}} \|\mathbf{u}\|^2_{L^3(Q_{R/2})} \lesssim R^{-2} \|\mathbf{u}\|^6_{L^{2,\infty}(Q_R)} + \|\mathbf{u}\|^2_{L^{2,\infty}(Q_R)}. \tag{3.11}$$

Finally, multiply this inequality with $(R/2)^{-1}$ to conclude the proof. ■

Remark 3.2. Other Caccioppoli inequalities for local suitable weak solutions of the Navier–Stokes equations were proved in [31] by essentially the same method. ♢

4 Local boundedness

In this section, based on the Caccioppoli inequality from the previous section we discuss and prove the announced local L^∞-bound for local suitable weak solutions. As already mentioned in the introduction, the required smallness hypothesis is in our case encoded by the rescaled maximal kinetic energy.

4.1 Main results

Theorem 4.1. *Let* $\mathbf{u}$ *be a local suitable weak solution of the Navier–Stokes equations* (1.1) *in the space-time cylinder* $Q = \Omega \times (a,b)$. *Then, there exist absolute constants* $\varepsilon > 0$ *and* $K > 0$ *such that for all* $Q_R = Q_R(x_0, t_0) \subset Q$ *it holds*

$$\frac{1}{R}\|\mathbf{u}\|^2_{L^{2,\infty}(Q_R)} \le \varepsilon^2 \quad \Longrightarrow \quad \|\mathbf{u}\|^2_{L^\infty(Q_{3R/4})} \le \frac{K}{R^2}\left(\frac{1}{R}\|\mathbf{u}\|^2_{L^{2,\infty}(Q_R)}\right). \tag{4.1}$$

The proposition below already contains the main technical part for the proof of this theorem. In this context, note first that it suffices to consider the case of the unit parabolic cylinder $Q = Q_1(0,0)$ since the general case then follows from a translation of the coordinate system and the scaling symmetry exhibited by the Navier–Stokes equations.

Let us also introduce some notation:

$$r_n = 2^{-n}, \quad I^n = (-r_n^2, 0), \quad B^n = B_{r_n}(0), \quad Q^n = B^n \times I^n, \quad n \ge 1.$$

Furthermore, we fix the specific local decomposition of the pressure (and we will stick with this choice during the whole section):

$$B = B^1, \quad p_h = p_{h,B}, \quad p_1 = p_{1,B}, \quad p_2 = p_{2,B}, \quad \mathbf{v} = \mathbf{u} + \nabla p_h.$$

With this notation at hand, the already mentioned proposition reads as follows.

Proposition 4.2. *Let* $\mathbf{u}$ *be a local suitable weak solution to* (1.1) *in the unit parabolic cylinder* $Q_1(0,0)$. *There exist absolute constants* $\varepsilon_* > 0$ *and* $k_* > 0$ *such that the condition*

$$\widetilde{C}(1) := \|\mathbf{u}\|^3_{L^{2,\infty}(Q_1(0,0))} \le \varepsilon_*^3$$

implies that for all $n \ge 2$ *the following two inequalities hold true*

$$r_n^{-5} \int_{Q^n} |\mathbf{u}|^3 \,\mathrm{d}x\mathrm{d}t \le k_*^{\frac{3}{2}} \widetilde{C}(1), \tag{4.2}$$

$$r_n^{-3} \operatorname*{ess\,sup}_{I^n} \int_{B^n} |\mathbf{v}|^2 \,\mathrm{d}x + r_n^{-3} \int_{Q^n} |\nabla \mathbf{v}|^2 \,\mathrm{d}x\mathrm{d}t \le k_* \widetilde{C}(1)^{\frac{2}{3}}. \tag{4.3}$$

4.2 On the idea and structure of the proof

Before we proceed with the proof of the proposition, let us give some remarks. Note first that Proposition 4.2 (and Theorem 4.1 as well) does not contain any explicit additional smallness assumption for the pressure field. This does not mean of course that our method works without the need for appropriate decay estimates of the pressure terms. But the specific nature of the local decomposition of the pressure field and the regularity properties of the operators $P_{q,U}$ (i.e. the Cattabriga–Solonnikov estimates) dovetail nicely with the $L^{2,\infty}$-norm of the velocity field. We already witnessed this in the proof of the Caccioppoli inequality (3.3). This is in contrast to the work in [1], where the inductive argument heavily relies on an explicit a priori smallness assumption for the pressure field.

Let us also comment on the idea and structure of the proof for Proposition 4.2, which basically consists of an inductive argument in the spirit of [1]. The induction basis obviously is given by $(4.2)_{n=2}$ and $(4.3)_{n=2}$. The latter one will be a consequence of our Caccioppoli inequality (3.3). The other one then follows by interpolation (i.e. inequality (A.10)). For $n \geq 3$, the induction step can be summarized schematically as follows:

$$\begin{aligned} (4.2)_m,\ m \in \{2,\dots,n-1\} \quad &\Rightarrow \quad (4.3)_n \\ (4.3)_n, \quad &\Rightarrow \quad (4.2)_n. \end{aligned}$$

The latter implication will again be a (more or less immediate) consequence of interpolation inequality (A.10). The main technical step of the proof is rather given by the first implication. For this one, we will make use of the generalized energy inequality (2.15) with respect to the local pressure decomposition introduced above.

But to make this approach work, one relies on a clever choice for a test function. For example, if one just takes an appropriate version of the test function used in the proof of the Caccioppoli inequality (3.3), one can merely expect a bound for the terms

$$\int_{Q^n} |\mathbf{u}|^3 \,\mathrm{d}x\mathrm{d}t, \quad \operatorname*{ess\,sup}_{I^n} \int_{B^n} |\mathbf{v}|^2 \,\mathrm{d}x + \int_{Q^n} |\nabla \mathbf{v}|^2 \,\mathrm{d}x\mathrm{d}t.$$

But in view of (4.2) and (4.3), we rather want to control the corresponding local averages of these terms. How can we achieve this? The main observation in this context is that on the right-hand side of the generalized energy inequality (2.15) the backward heat equation appears, and that a suitable smoothed version of the fundamental solution exactly obeys the required decay properties in order to introduce averages. Of course, this idea is not new, and in fact the precise construction of the test function is borrowed from the celebrated paper by Caffarelli et al. [1]. More explicitly, the construction is carried out as follows.

Let χ denote a suitable cut-off function, which is smooth at least on $\mathbb{R}^3 \times (-\infty, 0]$, and such that $0 \le \chi \le 1$, $\chi \equiv 1$ in Q^3 and $\chi \equiv 0$ outside of $Q_{1/6}$. Furthermore, we write ψ_n for the fundamental solution of the backward heat equation with singularity at $(0, r_n^2)$, i.e.

$$\psi_n(x,t) = \frac{c_0}{(r_n^2 - t)^{\frac{3}{2}}} \exp\left(-\frac{|x|^2}{4(r_n^2 - t)}\right), \quad (x,t) \in \mathbb{R}^3 \times (-\infty, r_n^2). \tag{4.4}$$

Put $\phi_n = \chi\psi_n \ge 0$. Clearly, ϕ_n solves the backward heat equation in Q^3. For reader's convenience, we also repeat from [1] the main estimates for the test function ϕ_n:

$$\left|\partial_t \phi_n + \Delta\phi_n\right| \le c \quad \text{in} \quad Q^1, \tag{4.5}$$

$$\frac{1}{c} r_n^{-3} \le \phi_n \le c r_n^{-3} \quad \text{and} \quad |\nabla\phi_n| \le c r_n^{-4} \quad \text{in} \quad Q^n, \tag{4.6}$$

$$\phi_n \le c r_k^{-3} \quad \text{and} \quad |\nabla\phi_n| \le c r_k^{-4} \quad \text{in} \quad Q^{k-1} \backslash Q^k, \quad 2 < k \le n. \tag{4.7}$$

Here, $c > 0$ is an absolute constant which is independent of n. The first set of inequalities in (4.6) then enables us to find a bound for the term

$$r_n^{-3} \operatorname*{ess\,sup}_{I^n} \int_{B^n} |\mathbf{v}|^2 \,\mathrm{d}x + r_n^{-3} \int_{Q^n} |\nabla\mathbf{v}|^2 \,\mathrm{d}x\mathrm{d}t,$$

if one inserts the test function ϕ_n in the generalized local energy inequality (2.15). The remaining work then consists of deriving appropriate estimates for each of the terms occurring on the right-hand side of (2.15). In view of our bounds in (4.6) and (4.7), for most of these terms we have to decompose the corresponding integral into a sum of integrals over the respective "dyadic shells" $Q^{k-1} \backslash Q^k$. This procedure works indeed, because in this sum one part of each summand can always be dealt with by the induction hypothesis (and also leaves a "small" quantity in terms of ε_*) and the remaining part consists of a positive power of the r_k's which is summable.

As it turns out, the most troublesome term in this context is given by the term involving p_1. This is due to the fact that the other two terms in the local pressure decomposition are harmonic, i.e. they can be treated by means of the powerful theory of harmonic functions. Nevertheless, based on the result for the bi-harmonic equation given in Proposition A.6, it is possible to derive estimates for p_1 which are good enough for our purposes. This is undoubtedly one of the major technical steps in the proof of Proposition 4.2, and the idea to master it successfully was suggested to the author by J. Wolf.

The final remark is about the constants which will appear throughout the induction. As in the proof of our Caccioppoli inequality (3.3), we will suppress unimportant proportionality constants as we will make use of the symbol $\lesssim$ until the very final step. In this regard, one only has to take care that these constants do not depend on n, because this will then enable us to choose the constants k_* and ε_* without being at risk to run into a circular argument.

4.3 Proof of the local L^∞-bound

Let us begin with the proof of Proposition 4.2.

Proof. We assume right from the beginning that $\varepsilon_* \le 1 \le k_*$ and proceed by induction. As already said above, the admissible values of the constants ε_* and k_* will be determined throughout the proof. Let us start with the

Induction basis ($n = 2$)**:** We immediately obtain due to our Caccioppoli inequality (3.3) that

$$\operatorname*{ess\,sup}_{I^2} \int\limits_{B^2} |\mathbf{u}|^2 \,\mathrm{d}x + \int\limits_{Q^2} |\nabla \mathbf{u}|^2 \,\mathrm{d}x\mathrm{d}t \lesssim \widetilde{C}(1)^{\frac{2}{3}}. \tag{4.8}$$

In addition, we can estimate by means of the classical Caccioppoli inequality for harmonic functions

$$\begin{aligned}\operatorname*{ess\,sup}_{I^2} \int\limits_{B^2} |\nabla p_h|^2 \,\mathrm{d}x + \int\limits_{Q^2} |\nabla^2 p_h|^2 \,\mathrm{d}x\mathrm{d}t &\lesssim \operatorname*{ess\,sup}_{I^1} \int\limits_{B^1} |\mathbf{u}|^2 \,\mathrm{d}x + \int\limits_{Q^1} |\nabla p_h|^2 \,\mathrm{d}x\mathrm{d}t \\ &\lesssim \widetilde{C}(1)^{\frac{2}{3}},\end{aligned}$$

hence there exists an absolute constant $C_1 \ge 1$ such that inequality $(4.3)_2$ holds with k_* replaced by C_1. Exploiting interpolation inequality (A.10) and inequality (4.8), we see that

$$\begin{aligned}\int\limits_{Q^2} |\mathbf{u}|^3 \,\mathrm{d}x\mathrm{d}t &\lesssim \left(\operatorname*{ess\,sup}_{I^2} \int_{B^2} |\mathbf{u}|^2 \,\mathrm{d}x\right)^{\frac{3}{4}} \left(\operatorname*{ess\,sup}_{I^2} \int_{B^2} |\mathbf{u}|^2 \,\mathrm{d}x + \int_{Q^2} |\nabla \mathbf{u}|^2 \,\mathrm{d}x\mathrm{d}t\right)^{\frac{3}{4}} \\ &\lesssim \left(\operatorname*{ess\,sup}_{I^2} \int_{B^2} |\mathbf{u}|^2 \,\mathrm{d}x + \int_{Q^2} |\nabla \mathbf{u}|^2 \,\mathrm{d}x\mathrm{d}t\right)^{\frac{3}{2}} \lesssim \widetilde{C}(1),\end{aligned}$$

i.e. inequality $(4.2)_2$ holds as well with k_* replaced by some suitable absolute constant $C_2 \ge 1$. Of course, we assume for all of what follows that $k_* \ge C_1 \vee C_2$.

Induction step ($n-1 \mapsto n$)**:** Now, consider $n \ge 3$. First note that obviously

$$r_n^{-5} \int\limits_{Q^n} |\mathbf{u}|^3 \,\mathrm{d}x\mathrm{d}t \le 2^5 r_{n-1}^{-5} \int\limits_{Q^{n-1}} |\mathbf{u}|^3 \,\mathrm{d}x\mathrm{d}t \le 2^5 k_*^{\frac{3}{2}} \widetilde{C}(1). \tag{4.9}$$

As already mentioned, we use ϕ_n (constructed as above) as a test function in (2.15) and obtain with (4.6) the following local energy estimate

$$r_n^{-3}\operatorname*{ess\,sup}_{I^n}\int\limits_{B^n}|\mathbf{v}|^2\,\mathrm{d}x+2r_n^{-3}\int\limits_{Q^n}|\nabla\mathbf{v}|^2\,\mathrm{d}x\mathrm{d}t\lesssim\sum_{l=1}^{6}J_l,\tag{4.10}$$

where we introduced the quantities

$$J_1=\int\limits_{Q^2}|\mathbf{v}|^2\left|\partial_t\phi+\Delta\phi_n\right|\mathrm{d}x\mathrm{d}t,\qquad J_2=\int\limits_{Q^2}|\mathbf{u}|^2|\mathbf{v}||\nabla\phi_n|\,\mathrm{d}x\mathrm{d}t,$$

$$J_3=\int_{I^2}\left|\int_{B^2}p_1\mathbf{v}\cdot\nabla\phi_n\,\mathrm{d}x\right|\mathrm{d}t,\qquad J_4=\int_{I^2}\left|\int_{B^2}p_2\mathbf{v}\cdot\nabla\phi_n\,\mathrm{d}x\right|\mathrm{d}t,$$

$$J_5=\int\limits_{Q^2}|\mathbf{u}\otimes\mathbf{u}||\nabla(\phi_n\nabla p_h)|\,\mathrm{d}x\mathrm{d}t,\quad J_6=\int\limits_{Q^2}|\mathbf{u}|^2|\nabla p_h||\nabla\phi_n|\,\mathrm{d}x\mathrm{d}t.$$

Now, we proceed with estimating term by term on the right hand side of our local energy inequality. Using (4.5), we start with

$$J_1\lesssim\int\limits_{Q^2}|\mathbf{u}|^2+|\nabla p_h|^2\,\mathrm{d}x\mathrm{d}t\lesssim\|\mathbf{u}\|^2_{L^{2,\infty}(Q^1)}\lesssim\widetilde{C}(1)^{\frac{2}{3}}.\tag{4.11}$$

For the second term, exploiting (4.6) and (4.7), we split the task as follows

$$\begin{aligned}J_2&\le\int\limits_{Q^2}|\mathbf{u}|^3|\nabla\phi_n|\,\mathrm{d}x\mathrm{d}t+\int\limits_{Q^2}|\mathbf{u}|^2|\nabla p_h||\nabla\phi_n|\,\mathrm{d}x\mathrm{d}t\\&\lesssim\sum_{k=2}^{n}r_k^{-4}\int\limits_{Q^k}|\mathbf{u}|^3\,\mathrm{d}x\mathrm{d}t+\sum_{k=2}^{n}r_k^{-4}\int\limits_{Q^k}|\mathbf{u}|^2|\nabla p_h|\,\mathrm{d}x\mathrm{d}t=:J_2'+J_2''.\end{aligned}$$

On one side, we obtain due to our induction hypothesis and (4.9) that

$$J_2'\lesssim k_*^{\frac{3}{2}}\widetilde{C}(1)\sum_{k=2}^{n}r_k\lesssim k_*^{\frac{3}{2}}\widetilde{C}(1).$$

On the other side, recall that ∇p_h is harmonic in B^1 for a.e. $t\in I^1$. Hence, by means of Hölder's inequality and the mean value inequality of harmonic functions, we deduce

$$\begin{aligned}J_2''&\lesssim\sum_{k=2}^{n}r_k\left(r_k^{-5}\int_{Q^k}|\mathbf{u}|^3\,\mathrm{d}x\mathrm{d}t\right)^{\frac{2}{3}}\left(r_k^{-5}\int_{Q^k}|\nabla p_h|^3\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{3}}\\&\lesssim k_*\widetilde{C}(1)^{\frac{2}{3}}\sum_{k=2}^{n}r_k\left(r_k^{-5}\int_{Q^k}\left|\operatorname*{ess\,sup}_{I^1}\|\nabla p_h\|_{L_1(B^1)}\right|^3\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{3}}\end{aligned}$$

$$\lesssim k_* \widetilde{C}(1)^{\frac{2}{3}} \operatorname*{ess\,sup}_{I^1} \left(\int_{B^1} |\nabla p_h|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}$$

$$\lesssim k_* \widetilde{C}(1)^{\frac{2}{3}} \|\mathbf{u}\|_{L^{2,\infty}(Q^1)} \lesssim k_* \widetilde{C}(1)^{\frac{2}{3}} \widetilde{C}(1)^{\frac{1}{3}} \lesssim k_* \widetilde{C}(1).$$

In particular, this entails the bound

$$J_2 \lesssim k_*^{3/2} \widetilde{C}(1), \tag{4.12}$$

and the same bound obviously holds true for J_6 as well:

$$J_6 \lesssim k_*^{3/2} \widetilde{C}(1). \tag{4.13}$$

For J_5, we estimate

$$J_5 \le \int\limits_{Q^2} |\mathbf{u}|^2 |\nabla p_h| |\nabla \phi_n| \, \mathrm{d}x\mathrm{d}t + \int\limits_{Q^2} |\mathbf{u}|^2 |\nabla^2 p_h| |\phi_n| \, \mathrm{d}x\mathrm{d}t$$

$$\lesssim k_* \widetilde{C}(1) + \sum_{k=2}^{n} r_k^2 \left(r_k^{-5} \int_{Q^k} |\mathbf{u}|^3 \, \mathrm{d}x\mathrm{d}t \right)^{\frac{2}{3}} \left(r_k^{-5} \int_{Q^k} |\nabla^2 p_h|^3 \, \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{3}}.$$

As $\nabla^2 p_h$ is harmonic in B^1 for a.e. $t \in I^1$ as well, we can work again with the mean value inequality to obtain together with the classical Cacciop- poli inequality that for all $2 \le k \le n$

$$\left(r_k^{-5} \int_{Q^k} |\nabla^2 p_h|^3 \, \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{3}} \lesssim \left(r_k^{-5} \int_{Q^k} \left| \operatorname*{ess\,sup}_{I^1} \|\nabla^2 p_h\|_{L_1(B_{1/3}(0))} \right|^3 \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{3}}$$

$$\lesssim \operatorname*{ess\,sup}_{I^1} \left(\int_{B_{1/3}(0)} |\nabla^2 p_h|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}$$

$$\lesssim \operatorname*{ess\,sup}_{I^1} \left(\int_{B^1} |\nabla p_h|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \lesssim \widetilde{C}(1)^{\frac{1}{3}}.$$

All in all, we infer that

$$J_5 \lesssim k_* \widetilde{C}(1) + k_* \widetilde{C}(1)^{\frac{2}{3}} \widetilde{C}(1)^{\frac{1}{3}} \lesssim k_* \widetilde{C}(1). \tag{4.14}$$

It remains to estimate the pressure terms. We begin with J_4 and recall that p_2 is harmonic in B^1 for a.e. $t \in I^1$. Note that $\mathbf{v}$ is a solenoidal field, hence arguing as in [1] yields the bound

$$J_4 \lesssim \sum_{k=3}^{n} r_k^{-4} \int\limits_{Q^k} |\mathbf{v}| |p_2 - [p_2]_{B^k}| \, \mathrm{d}x\mathrm{d}t + \int\limits_{Q^2} |\mathbf{v}| |p_2| \, \mathrm{d}x\mathrm{d}t =: J_4' + J_4''.$$

For the sake of completeness, we repeat the argument from [1]. The first step consists of choosing cut-off functions $\chi_k \in C_c^\infty(Q^k)$ such that it holds

$0 \leq \chi_k \leq 1$, $\chi_k \equiv 1$ in $Q_{3r_k/4}(0,0)$ and $|\nabla \chi_k| \leq c r_k^{-1}$, where $c > 0$ is an absolute constant which does not depend on $k \geq 1$. In a next step, we bound J_4 as follows

$$J_4 = \int_{I^2} \left| \int_{B^2} p_2 \mathbf{v} \cdot \nabla(\chi_1 \phi_n)\,\mathrm{d}x \right| \mathrm{d}t$$
$$\leq \sum_{k=1}^{n-1} \int_{I^2} \left| \int_{B^2} p_2 \mathbf{v} \cdot \nabla((\chi_k - \chi_{k+1})\phi_n)\,\mathrm{d}x \right| \mathrm{d}t + \int_{I^2} \left| \int_{B^2} p_2 \mathbf{v} \cdot \nabla(\chi_n \phi_n)\,\mathrm{d}x \right| \mathrm{d}t.$$

If $k \in \{1,2\}$ then we simply estimate

$$\int_{I^2} \left| \int_{B^2} p_2 \mathbf{v} \cdot \nabla((\chi_k - \chi_{k+1})\phi_n)\,\mathrm{d}x \right| \mathrm{d}t \lesssim \int\limits_{Q^2} |p_2||\mathbf{v}|\,\mathrm{d}x\mathrm{d}t.$$

In case of $k \in \{3,\ldots,n-1\}$, we can write

$$\int_{I^2} \left| \int_{B^2} p_2 \mathbf{v} \cdot \nabla((\chi_k - \chi_{k+1})\phi_n)\,\mathrm{d}x \right| \mathrm{d}t =$$
$$= \int_{I^k} \left| \int_{B^k} (p_2 - [p_2]_{B^k}) \mathbf{v} \cdot \nabla((\chi_k - \chi_{k+1})\phi_n)\,\mathrm{d}x \right| \mathrm{d}t.$$

In this step, we made use of the fact that $\mathbf{v}$ is a solenoidal field. In particular,

$$\int_{I^2} \left| \int_{B^2} p_2 \mathbf{v} \cdot \nabla((\chi_k - \chi_{k+1})\phi_n)\,\mathrm{d}x \right| \mathrm{d}t \lesssim r_k^{-4} \int\limits_{Q^k} |p_2 - [p_2]_{B^k}||\mathbf{v}|\,\mathrm{d}x\mathrm{d}t$$

for every $k \in \{3,\ldots,n-1\}$, and the same sort of reasoning yields the estimate

$$\int_{I^2} \left| \int_{B^2} p_2 \mathbf{v} \cdot \nabla(\chi_n \phi_n)\,\mathrm{d}x \right| \mathrm{d}t \lesssim r_n^{-4} \int\limits_{Q^n} |p_2 - [p_2]_{B^n}||\mathbf{v}|\,\mathrm{d}x\mathrm{d}t,$$

i.e. the asserted upper bound for J_4 does hold indeed.

Let us now proceed with the terms J_4' and J_4''. At first, we easily estimate with Hölder's and Young's inequality

$$J_4'' \lesssim \left(\int_{Q^1} |\mathbf{v}|^2\,\mathrm{d}x\mathrm{d}t \right)^{\frac{1}{2}} \left(\int_{Q^2} |p_2|^2\,\mathrm{d}x\mathrm{d}t \right)^{\frac{1}{2}}$$
$$\lesssim \left\{ \|\mathbf{u}\|^2_{L^{2,\infty}(Q^1)} + \|\nabla \mathbf{u}\|^2_{L^2(Q^1)} \right\} \lesssim \widetilde{C}(1)^{\frac{2}{3}}.$$

Note that in the last step we also exploited our Caccioppoli inequality (3.3). Next, we estimate

$$J_4' \lesssim \sum_{k=3}^{n} r_k \left(r_k^{-5} \int_{Q^k} |\mathbf{v}|^2\,\mathrm{d}x\mathrm{d}t \right)^{\frac{1}{2}} \left(r_k^{-5} \int_{Q^k} |p_2 - [p_2]_{B^k}|^2\,\mathrm{d}x\mathrm{d}t \right)^{\frac{1}{2}}.$$

On one side, we obtain for $3 \le k \le n$ by induction hypothesis, the mean value inequality for p_h and by (4.9) that

$$\Big(r_k^{-5}\int_{Q^k}|\mathbf{v}|^2\,\mathrm{d}x\mathrm{d}t\Big)^{\frac12} \lesssim \Big(r_k^{-5}\int_{Q^k}|\mathbf{u}|^3\,\mathrm{d}x\Big)^{\frac13} + \Big(r_k^{-5}\int_{Q^k}|\nabla p_h|^2\,\mathrm{d}x\mathrm{d}t\Big)^{\frac12} \lesssim k_*^{\frac12}\widetilde{C}(1)^{\frac13}.$$

On the other side, using the classical Campanato inequality (A.7) we obtain for all $3 \le k \le n$ the bound

$$\begin{aligned}\Big(r_k^{-5}\int_{Q^k}|p_2-[p_2]_{B^k}|^2\,\mathrm{d}x\mathrm{d}t\Big)^{\frac12} &\lesssim \Big(r_k^{-5}\int_{I^k} r_k^{3+2}\int_{B^1}|p_2-[p_2]_{B^1}|^2\,\mathrm{d}x\mathrm{d}t\Big)^{\frac12}\\ &\lesssim \Big(\int_{Q^1}|p_2-[p_2]_{B^1}|^2\,\mathrm{d}x\mathrm{d}t\Big)^{\frac12}\\ &\lesssim \|\nabla\mathbf{u}\|_{L^2(Q^1)} \lesssim \widetilde{C}(1)^{\frac13}.\end{aligned}$$

Therefore, the above inequalities result in the bound

$$J_4 \lesssim k_*^{\frac12}\widetilde{C}(1)^{\frac23}. \tag{4.15}$$

For the last term, we again start with

$$J_3 \lesssim \sum_{k=3}^{n} r_k^{-4}\int\limits_{Q^k}|\mathbf{v}||p_1-[p_1]_{B^k}|\,\mathrm{d}x\mathrm{d}t + \int\limits_{Q^2}|\mathbf{v}||p_1|\,\mathrm{d}x\mathrm{d}t =: J_3' + J_3''.$$

Similarly as above, we deduce from our Caccioppoli inequality (3.3) that

$$J_3'' \lesssim \Big(\int_{Q^1}|\mathbf{v}|^3\,\mathrm{d}x\mathrm{d}t\Big)^{\frac13}\Big(\int_{Q^2}|p_1|^{\frac32}\,\mathrm{d}x\mathrm{d}t\Big)^{\frac23} \lesssim \Big\{\|\mathbf{u}\|^2_{L^3(Q^1)} + \|\mathbf{u}\|^4_{L^3(Q^1)}\Big\} \lesssim \widetilde{C}(1)^{\frac23}.$$

Moreover, we infer that

$$J_3' \lesssim \sum_{k=3}^{n} r_k^{\frac13}\Big(r_k^{-5}\int_{Q^k}|\mathbf{v}|^3\,\mathrm{d}x\mathrm{d}t\Big)^{\frac13}\Big(r_k^{-4}\int_{Q^k}|p_1-[p_1]_{B^k}|^{\frac32}\,\mathrm{d}x\mathrm{d}t\Big)^{\frac23}.$$

The term involving $\mathbf{v}$ is again of no problem due to (4.9) and the induction hypothesis:

$$\Big(r_k^{-5}\int_{Q^k}|\mathbf{v}|^3\,\mathrm{d}x\mathrm{d}t\Big)^{\frac13} \lesssim \Big(r_k^{-5}\int_{Q^k}|\mathbf{u}|^3\,\mathrm{d}x\Big)^{\frac13} + \Big(r_k^{-5}\int_{Q^k}|\nabla p_h|^3\,\mathrm{d}x\mathrm{d}t\Big)^{\frac13} \lesssim k_*^{\frac12}\widetilde{C}(1)^{\frac13},$$

where $3 \le k \le n$. As already remarked, finding an appropriate upper bound for the other term is a little bit more delicate since p_1 is not harmonic, i.e. our techniques from above are not directly amenable. But as we will see, a suitable decomposition of the non-harmonic pressure p_1 together with an iterative procedure will help us out.

To this end, fix numbers $r_n \le \rho < r \le r_2$ and $t \in I^1$ such that it holds $p_1(t) \in L^{3/2}(B^1)$ and $\mathbf{u}(t) \in W^{1,2}_\sigma(B_1(0),\mathbb{R}^3) \cap L^2_\sigma(B_1(0),\mathbb{R}^3)$. By Proposition A.6 we can find a (uniquely determined) function $w_r(t) \in W^{2,3/2}_0(B_r)$, such that $\Delta^2 w_r(t) = \Delta p_1(t)$ holds in B_r in the sense of distributions. In other words, for every $\phi \in C^\infty_c(B_r)$ we have

$$\int_{B_r} \big(\Delta w_r(t) - p_1(t)\big)\Delta\phi \,\mathrm{d}x = 0.$$

Therefore, by means of Weyl's lemma we obtain for a.e. $t \in I^1$ the decomposition

$$p_1(t) - [p_1(t)]_{B_r} = \Delta w_r(t) + q_r(t),$$

where $q_r(t) \in L^{3/2}(B_r)$ is a harmonic function. On the other side, it is straightforward to show that for all $\phi \in C^\infty_c(B_r)$

$$\int_{B_r} \Delta w_r(t)\Delta\phi \,\mathrm{d}x = \int_{B_r} \big(\mathbf{u}(t)\otimes\mathbf{u}(t)\big) : \nabla^2\phi \,\mathrm{d}x,$$

i.e. $w_r(t)$ is a solution to the bi-harmonic equation

$$\Delta^2 w_r(t) = \operatorname{div}\operatorname{div}\big(\mathbf{u}(t)\otimes\mathbf{u}(t)\big)$$

in the sense of distributions. In particular, the following important inequality holds true due to (A.11) and the uniqueness statement of Proposition A.6

$$\|\Delta w_r(t)\|_{L^{\frac{3}{2}}(B_r)} \lesssim \|\mathbf{u}(t)\otimes\mathbf{u}(t)\|_{L^{\frac{3}{2}}(B_r)}.$$

It is worth noticing at this point that the proportionality constant implicit in this inequality does not depend on the radius r.

With the help of this decomposition, we can now proceed as follows

$$\begin{aligned}
&\int_{B_\rho} \big|p_1(t) - [p_1(t)]_{B_\rho}\big|^{\frac{3}{2}} \,\mathrm{d}x \\
&\qquad = \int_{B_\rho} \big|p_1(t) - [p_1(t)]_{B_r} - \big[p_1(t) - [p_1(t)]_{B_r}\big]_{B_\rho}\big|^{\frac{3}{2}} \,\mathrm{d}x \\
&\qquad \lesssim \int_{B_\rho} \big|\Delta w_r(t) - [\Delta w_r(t)]_{B_\rho}\big|^{\frac{3}{2}} \,\mathrm{d}x + \int_{B_\rho} \big|q_r(t) - [q_r(t)]_{B_\rho}\big|^{\frac{3}{2}} \,\mathrm{d}x \\
&\qquad \lesssim \|\mathbf{u}(t)\|^3_{L^3(B_r)} + \int_{B_\rho} \big|q_r(t) - [q_r(t)]_{B_\rho}\big|^{\frac{3}{2}} \,\mathrm{d}x.
\end{aligned}$$

As $q_r(t)$ is harmonic in B_r, we can estimate due to the generalized L^p-Campanato inequality from Proposition A.3

$$\int\limits_{B_\rho} \left|q_r(t)-[q_r(t)]_{B_\rho}\right|^{\frac{3}{2}}\,\mathrm{d}x \lesssim \left(\frac{\rho}{r}\right)^{3+\frac{3}{2}} \int\limits_{B_r} \left|q_r(t)-[q_r(t)]_{B_r}\right|^{\frac{3}{2}}\,\mathrm{d}x$$
$$\lesssim \left(\frac{\rho}{r}\right)^{\frac{9}{2}} \int\limits_{B_r} \left|p_1(t)-[p_1(t)]_{B_r}\right|^{\frac{3}{2}}\,\mathrm{d}x.$$

As we can surely find $k \in \{2,\dots,n-1\}$ such that $r_{k+1} < r \le r_k$, we may infer from the induction hypothesis that

$$\|\mathbf{u}\|^3_{L^3(Q_r)} \le r^4\left(r^{-5}\|\mathbf{u}\|^3_{L^3(Q_r)}\right) \lesssim r^4\left(r_k^{-5}\|\mathbf{u}\|^3_{L^3(Q^k)}\right) \lesssim k_*^{\frac{3}{2}}\widetilde{C}(1)r^4.$$

Integrating in time and then putting all estimates together, we therefore arrive at

$$\int\limits_{Q_\rho} \left|p_1-[p_1]_{B_\rho}\right|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t \lesssim k_*^{\frac{3}{2}}\widetilde{C}(1)r^4 + \left(\frac{\rho}{r}\right)^{\frac{9}{2}} \int\limits_{Q_r} \left|p_1-[p_1]_{B_r}\right|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t.$$

This bound holds for all $r_n \le \rho < r \le r_2$. Hence, we are in position to iterate this inequality, i.e. by Proposition A.2 it is also true that the following bound is satisfied for all $r_n \le \rho < r \le r_2$

$$\int\limits_{Q_\rho} \left|p_1-[p_1]_{B_\rho}\right|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t \lesssim k_*^{\frac{3}{2}}\widetilde{C}(1)\rho^4 + \left(\frac{\rho}{r}\right)^{4} \int\limits_{Q_r} \left|p_1-[p_1]_{B_r}\right|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t.$$

In particular, this entails for all $3 \le k \le n$ that

$$r_k^{-4}\int_{Q^k} |p_1-[p_1]_{B^k}|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t \lesssim k_*^{\frac{3}{2}}\widetilde{C}(1) + \int\limits_{Q^2} |p_1|^{\frac{3}{2}}\,\mathrm{d}x\mathrm{d}t$$
$$\lesssim k_*^{\frac{3}{2}}\widetilde{C}(1) + \|\mathbf{u}\|^3_{L^3(Q^1)} \lesssim k_*^{\frac{3}{2}}\widetilde{C}(1),$$

hence $J_3' \lesssim k_*^{\frac{3}{2}}\widetilde{C}(1)$. All in all, this then leads to the bound

$$J_3 \lesssim k_*^{\frac{3}{2}}\widetilde{C}(1) + \widetilde{C}(1)^{\frac{2}{3}}. \tag{4.16}$$

Eventually, the bounds from (4.11) to (4.16) give rise to

$$r_n^{-3}\operatorname*{ess\,sup}_{I^n}\int\limits_{B^n} |\mathbf{v}|^2\,\mathrm{d}x + r_n^{-3}\int\limits_{Q^n} |\nabla\mathbf{v}|^2\,\mathrm{d}x\mathrm{d}t \le c_1\left(k_*^{\frac{1}{2}}\varepsilon_* + k_*^{-\frac{1}{2}}\right)k_*\widetilde{C}(1)^{\frac{2}{3}}.$$

Using interpolation inequality (A.10), this also means that

$$r_n^{-5}\int\limits_{Q^n} |\mathbf{v}|^3\,\mathrm{d}x\mathrm{d}t \le c_2c_1^{\frac{3}{2}}\left(k_*^{\frac{1}{2}}\varepsilon_* + k_*^{-\frac{1}{2}}\right)^{\frac{3}{2}} k_*^{\frac{3}{2}}\widetilde{C}(1).$$

Since we can estimate again by mean value inequality to obtain

$$r_n^{-5} \int_{Q^n} |\nabla p_h|^3 \, \mathrm{d}x\mathrm{d}t \le c_2 r_n^{-5} \int_{Q^n} \left| \operatorname*{ess\,sup}_{I^1} \|\nabla p_h\|_{L^1(B^1)} \right|^3 \mathrm{d}x\mathrm{d}t$$
$$\le c_2 \operatorname*{ess\,sup}_{I^1} \|\nabla p_h\|^3_{L^2(B^1)}$$
$$\le c_2 \operatorname*{ess\,sup}_{I^1} \|\mathbf{u}\|^3_{L^2(B^1)} \le c_2 \widetilde{C}(1),$$

we also find that

$$r_n^{-5} \int_{Q^n} |\mathbf{u}|^3 \, \mathrm{d}x\mathrm{d}t \le c_2 c_1^{\frac{3}{2}} \left(k_*^{\frac{1}{2}} \varepsilon_* + k_*^{-\frac{1}{2}} \right)^{\frac{3}{2}} k_*^{\frac{3}{2}} \widetilde{C}(1).$$

We emphasize the fact that c_1 and c_2 are absolute constants which in particular neither depend on k_* nor n. Thus, choosing

$$\widetilde{c}_2 = c_2^{\frac{2}{3}}, \quad k_* \ge (2c_1\widetilde{c}_2)^2, \quad \varepsilon_* \le \frac{1}{(2c_1\widetilde{c}_2)^2}, \tag{4.17}$$

yields the desired bounds and the result follows. ■

We now turn to the proof of Theorem 4.1. Everything what remains to do is to exploit the scaling symmetry of local suitable weak solutions in an appropriate manner.

Proof. Consider some arbitrary space-time point $(y_0, s_0) \in Q_{3/4}(0,0)$ and let

$$\widetilde{\mathbf{u}}(x,t) = 4^{-1}\mathbf{u}\left(y_0 + 4^{-1}x, s_0 + 4^{-2}t\right)$$

for $(x,t) \in Q_1(0,0)$. Then, $\widetilde{\mathbf{u}}$ is a local suitable weak solution for the Navier–Stokes system in $Q_1(0,0)$ and

$$\|\widetilde{\mathbf{u}}\|^3_{L^{2,\infty}(Q_1(0,0))} = 4^{\frac{3}{2}} \|\mathbf{u}\|^3_{L^{2,\infty}(Q_{1/4}(y_0,s_0))} \le 4^{\frac{3}{2}} \widetilde{C}(1).$$

This motivates to choose $\varepsilon = 4^{-1/2}\varepsilon_*$, as the above proposition applied to the velocity $\widetilde{\mathbf{u}}$ would then imply for all $n \ge 2$ the inequality

$$r_n^{-5} \int_{Q^n} |\widetilde{\mathbf{u}}(x,t)|^3 \, \mathrm{d}x\mathrm{d}t \le k_*^{\frac{3}{2}} \|\widetilde{\mathbf{u}}\|^3_{L^{2,\infty}(Q_1(0,0))} \le (4k_*)^{\frac{3}{2}} \widetilde{C}(1).$$

But there is also a constant $c > 0$ such that

$$r_n^{-5} \int_{Q^n} |\widetilde{\mathbf{u}}(x,t)|^3 \, \mathrm{d}x\mathrm{d}t = \frac{c^{-\frac{3}{2}}}{|Q^{n+1}(y_0,s_0)|} \int_{Q^{n+1}(y_0,s_0)} |\mathbf{u}(y,s)|^3 \, \mathrm{d}y\mathrm{d}s.$$

Now, we choose $K = 4ck_*$ and deduce that for all $n \ge 3$

$$\frac{1}{|Q^n(y_0,s_0)|} \int_{Q^n(y_0,s_0)} |\mathbf{u}(y,s)|^3 \, \mathrm{d}y\mathrm{d}s \le K^{\frac{3}{2}} \widetilde{C}(1). \tag{4.18}$$

As almost every $(y_0, s_0) \in Q_{1/2}(0,0)$ is a Lebesgue point of $\mathbf{u}$, passing to the limit in inequality (4.18) implies

$$\|\mathbf{u}\|^2_{L^\infty(Q_{3/4}(0,0))} \le K\|\mathbf{u}\|^2_{L^{2,\infty}(Q_1(0,0))}. \tag{4.19}$$

As remarked above, this is all what is needed in order to derive the general statement of our theorem. ■

5 Estimates for spatial derivatives

In this section, we want to derive L^∞-bounds for spatial derivatives of local suitable weak solutions to the Navier–Stokes equations (1.1). The main result in this regard reads as follows:

Theorem 5.1. *Let $\mathbf{v}$ be a local suitable weak solution to the Navier–Stokes equations in the unit parabolic cylinder Q_1. There exist absolute constants $\varepsilon_1 > 0$ and $K_1 > 0$, such that*

$$\|\mathbf{v}\|^2_{L^{2,\infty}(Q_1)} \le \varepsilon_1 \quad \Rightarrow \quad \|\nabla \mathbf{v}\|^2_{L^\infty(Q_{1/2})} \le K_1 \|\mathbf{v}\|^2_{L^{2,\infty}(Q_1)}. \tag{5.1}$$

5.1 Local regularity for a Navier–Stokes type system

In order to prove this result, we will consider first the following abstract setting: Let $\Omega \subset \mathbb{R}^3$ be a domain, $I = (a,b) \subset \mathbb{R}$ a bounded interval and let $Q = \Omega \times I$ be the associated space-time cylinder. Furthermore, let $\mathbf{A}\colon Q \to \mathbb{R}^3$ be a solenoidal field which is measurable and bounded. In the following, we study the local regularity of a certain type of weak solutions for the system

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + \operatorname{div}(\mathbf{u} \otimes \mathbf{A}) = -\nabla p & \text{in } Q, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } Q. \end{cases} \tag{5.2}$$

As we are again solely interested in interior local regularity properties, we do not deal with any sort of boundary and/or initial data for this system. Consequently, weak solutions for this system are defined in analogy to Definition 1.1.

What is the motivation to study this system? For example, assume that $(\mathbf{v}, \pi)$ is a weak solution to the Navier–Stokes system in the unit parabolic cylinder $Q_1 = B_1(0) \times (-1, 0)$

$$\begin{cases} \partial_t \mathbf{v} - \Delta \mathbf{v} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) = -\nabla \pi, \\ \operatorname{div} \mathbf{v} = 0. \end{cases} \tag{5.3}$$

Furthermore, assume that the velocity field $\mathbf{v}$ is bounded in Q_1 (as it is the case at least in $Q_{1/2}$ in the situation of Theorem 4.1). Now, at least on a formal level, each (spatial) partial derivative $\partial_k \mathbf{v}$ of the velocity field satisfies the following system in Q_1

$$\begin{cases} \partial_t \partial_k \mathbf{v} - \Delta \partial_k \mathbf{v} + \operatorname{div}(\mathbf{v} \otimes \partial_k \mathbf{v}) + \operatorname{div}(\partial_k \mathbf{v} \otimes \mathbf{v}) = -\nabla \partial_k \pi, \\ \operatorname{div} \partial_k \mathbf{v} = 0. \end{cases}$$

Thus, we are almost in the situation of (5.2), except for a symmetrization of the tensor product.

Theorem 5.2. *Let* $\mathbf{u}$ *be a local suitable weak solution to* (5.2) *in the unit parabolic cylinder* Q_1*. Then, there exists a constant* $K > 0$ *such that*

$$\|\mathbf{u}\|^2_{L^\infty(Q_{1/2})} \le K\|\mathbf{u}\|^2_{L^{2,\infty}(Q_1)}. \tag{5.4}$$

The constant K *just depends on the quantity* $1 \vee \|\mathbf{A}\|_{L^\infty(Q_1)}$*.*

At this point, we have not said anything about what we mean by a "local suitable weak solution" for the system (5.2). Given our discussion in Section 2, the following definition is not surprising at all.

Definition 5.3. Let $\mathbf{u}$ be a weak solution to (5.2) in the space-time cylinder $\Omega \times (a, b)$. We call $\mathbf{u}$ a local suitable weak solution, if for every ball $B \subset\subset \Omega$, every non-negative test function $\phi \in C_c^\infty(B \times (a, b))$ and for a.e. $t \in (a, b)$ the following inequality holds true

$$\begin{aligned}
&\int_B |\mathbf{v}_B(t)|^2 \phi(t)\,dx + 2\int_a^t \int_B |\nabla \mathbf{v}_B|^2 \phi\,dxds \le \\
&\le \int_a^t \int_B |\mathbf{v}_B|^2 (\partial_t \phi + \Delta\phi)\,dxds + 2\int_a^t \int_B p_{0,B} \mathbf{v}_B \cdot \nabla\phi\,dxds + \\
&\qquad + 2\int_a^t \int_B (\mathbf{u} \otimes \mathbf{A}) : (\phi \nabla \mathbf{v}_B + \mathbf{v}_B \otimes \nabla\phi)\,dxds.
\end{aligned} \tag{5.5}$$

Again, we will call this bound a generalized local energy inequality for the system (5.2). This time, the individual terms in the local pressure decomposition are

$$\begin{aligned}
p_{h,B} &:= -\mathscr{P}_{2,B}\mathbf{u} = -\mathscr{P}_{2,B}\mathbf{u}, \\
p_{1,B} &:= -\mathscr{P}_{2,B}\operatorname{div}(\mathbf{u} \otimes \mathbf{A}), \\
p_{2,B} &:= \mathscr{P}_{2,B}\Delta\mathbf{u} = \mathscr{P}_{2,B}\Delta\mathbf{u}.
\end{aligned}$$

We also put

$$\mathbf{v}_B := \mathbf{u} + \nabla p_{h,B}, \quad p_{0,B} := p_{2,B} - p_{1,B}. \qquad \diamondsuit$$

Example 5.4. Let $\mathbf{v}$ be a weak solution to the Navier–Stokes equations (5.3) in the unit parabolic cylinder Q_1. Let us assume furthermore that $\mathbf{v}$ is bounded in a neighbourhood of $Q_{1/2}$ in Q_1, say $\mathbf{v} \in L^\infty(Q_{3/4})$.

Then, for each $i \in \{1, 2, 3\}$, the partial derivative $\mathbf{u} = \partial_i \mathbf{v}$ is a local suitable weak solution to (5.2) in $Q_{1/2}$ (with respect to $\mathbf{A} = \mathbf{v}$ and the obvious modification $\operatorname{div}(\mathbf{u} \otimes \mathbf{A}) \mapsto 2\operatorname{div}(\mathbf{u} \otimes_s \mathbf{A})$ for both the system and the corresponding notion of a local suitable weak solution).

Indeed: Consider $B \subset\subset B_{3/4}$. The strategy is to proceed by the difference quotient method. So let us write $\Delta_k^i \mathbf{v} = k^{-1}(\mathbf{v}(\cdot + ke_i) - \mathbf{v})$ for small enough $k \ne 0$ and $\tau_k^i \mathbf{A} = \mathbf{A}(\cdot + ke_i)$. (To be precise, the introduced translation and difference quotient operators only act on the spatial variable and e_i denotes the i-th standard basis vector in $\mathbb{R}^3$.)

As $\mathbf{v}$ is a weak solution to the Navier–Stokes equations, the following relation holds for every test function $\varphi \in C_c^1\big(-1/4,0;C_{c,\sigma}^\infty(B,\mathbb{R}^3)\big)$ and small enough $k \neq 0$

$$-\int_{-\frac14}^0\int_B \Delta_k^i\mathbf{v}\partial_t\varphi\,dx\,dt - \int_{-\frac14}^0\int_B\big(\Delta_k^i\mathbf{v}\otimes\mathbf{A}+\tau_k^i\mathbf{A}\otimes\Delta_k^i\mathbf{v}\big)\colon\nabla\varphi\,dx\,dt + \\ + \int_{-\frac14}^0\int_B \nabla\Delta_k^i\mathbf{v}\colon\nabla\varphi\,dx\,dt = 0.$$

Along the same lines as in [31], we infer from this using Steklov averages that for every $\phi \in C_c^\infty(B\times(-1/4,0))$ and for a.e. $t\in(-1/4,0)$

$$\int_B\big|\Delta_k^i\mathbf{v}(t)+\nabla p_{h,B}^{k,i}(t)\big|^2\phi(t)\,dx - \int_{-\frac14}^t\int_B\big|\Delta_k^i\mathbf{v}+\nabla p_{h,B}^{k,i}\big|^2\partial_t\phi\,dx\,ds \\ +\int_{-\frac14}^t\int_B\nabla\big(\Delta_k^i\mathbf{v}+\nabla p_{h,B}^{k,i}\big)\colon\nabla\varphi\,dx\,ds - \int_{-\frac14}^t\int_B\big(p_{1,B}^{k,i}+p_{2,B}^{k,i}\big)\operatorname{div}\varphi\,dx\,ds \\ -\int_{-\frac14}^t\int_B\big(\Delta_k^i\mathbf{v}\otimes\mathbf{A}+\tau_k^i\mathbf{A}\otimes\Delta_k^i\mathbf{v}\big)\colon\nabla\varphi\,dx\,ds = 0,$$

where $\varphi = 2\phi\big(\Delta_k^i\mathbf{v}+\nabla p_{h,B}^{k,i}\big)$ and

$$p_{h,B}^{k,i} = -\mathscr{P}_{2,B}\Delta_k^i\mathbf{v},\quad p_{2,B}^{k,i} = \mathscr{P}_{2,B}\Delta(\Delta_k^i\mathbf{v}), \\ p_{1,B}^{k,i} = -\mathscr{P}_{2,B}\big(\operatorname{div}\big(\Delta_k^i\mathbf{v}\otimes\mathbf{A}\big)+\operatorname{div}\big(\tau_k^i\mathbf{A}\otimes\Delta_k^i\mathbf{v}\big)\big).$$

Let us abbreviate for what follows $\mathbf{w}_B^{k,i} = \Delta_k^i\mathbf{v}+\nabla p_{h,B}^{k,i}$. Integration by parts then reveals that for a.e. $t\in(-1/4,0)$ and all $\phi\in C_c^\infty(B\times(-1/4,0))$ the following energy equality does hold

$$\int_B\big|\mathbf{w}_B^{k,i}(t)\big|^2\phi(t)\,dx + 2\int_{-\frac14}^t\int_B\big|\nabla\mathbf{w}_B^{k,i}\big|^2\phi\,dx\,ds = \\ = \int_{-\frac14}^t\int_B\big|\mathbf{w}_B^{k,i}\big|^2(\partial_t\phi+\Delta\phi)\,dx\,ds + 2\int_{-\frac14}^t\int_B\big(p_{1,B}^{k,i}+p_{2,B}^{k,i}\big)\mathbf{w}_B^{k,i}\cdot\nabla\phi\,dx\,ds \\ +2\int_{-\frac14}^t\int_B\big(\Delta_k^i\mathbf{v}\otimes\mathbf{A}+\tau_k^i\mathbf{A}\otimes\Delta_k^i\mathbf{v}\big)\colon\big(\phi\nabla\mathbf{w}_B^{k,i}+\mathbf{w}_B^{k,i}\otimes\nabla\phi\big)\,dx\,ds.$$

In the sequel, we use this equality for two purposes. In a first step, we exploit it in order to derive higher regularity for the velocity field. In the second step, we then establish the asserted energy inequality.

To this end, note first that $\mathbf{w}_B^{k,i} = \Delta_k^i(\mathbf{v}+\nabla p_{h,B})$. Also recall that

$$\nabla p_{h,B} \in L^2\big(-1,0;W^{1,2}(B,\mathbb{R}^3)\big)$$

due to the Cattabriga–Solonnikov estimates. Hence, it holds

$$\big\|\Delta_k^i\mathbf{v}\big\|_{L^2(B\times(-1,0))} \vee \big\|\mathbf{w}_B^{k,i}\big\|_{L^2(B\times(-1,0))} \lesssim 1, \tag{5.6}$$

and this bound is uniform over small enough $k \neq 0$. Using the boundedness of the velocity field and again the Cattabriga–Solonnikov estimates, a similar bound also holds for the other pressure terms, i.e. uniformly over small enough $k \neq 0$

$$\|p_{1,B}^{k,i}\|_{L^2(B\times(-1,0))} \vee \|p_{2,B}^{k,i}\|_{L^2(B\times(-1,0))} \lesssim 1. \tag{5.7}$$

In addition, using Hölder's and Young's inequality we obtain the bound

$$\begin{aligned} &2\int_{-\frac{1}{4}}^{t}\int_B \big(\Delta_k^i \mathbf{v}\otimes\mathbf{A} + \tau_k^i \mathbf{A}\otimes\Delta_k^i\mathbf{v}\big) : \phi\nabla\mathbf{w}_B^{k,i}\,dxds \\ &\qquad \leq C\|\Delta_k^i\mathbf{v}\|_{L^2(B\times(-1,0))} + \int_{-\frac{1}{4}}^{t}\int_B \big|\nabla\mathbf{w}_B^{k,i}\big|^2\phi\,dxds. \end{aligned} \tag{5.8}$$

for some constant $C > 0$. Fortunately, the last term appears with a factor 2 on the left-hand side of our equality above. For this reason, we may subtract it and obtain for a.e. $t \in (-1/4, 0)$ and all $\phi \in C_c^\infty(B \times (-1/4, 0))$

$$\int_B \big|\mathbf{w}_B^{k,i}(t)\big|^2\phi(t)\,dx + \int_{-\frac{1}{4}}^{t}\int_B \big|\nabla\mathbf{w}_B^{k,i}\big|^2\phi\,dxds \lesssim 1. \tag{5.9}$$

(Of course, the proportionality constant implicit in this bound does depend on the test function.) Finally, note that $\nabla\mathbf{w}_B^{k,i} = \Delta_k^i(\nabla\mathbf{v} + \nabla^2 p_{h,B})$. We infer from this by standard theory for difference quotient approximations that

$$\nabla\mathbf{v} \in L^\infty\big(-1/4, 0; L^2(B_{1/2}, \mathbb{R}^{3\times 3})\big) \cap L^2\big(-1/4, 0; W^{1,2}(B_{1/2}, \mathbb{R}^{3\times 3})\big). \tag{5.10}$$

In particular, $\nabla^2\mathbf{v} \in L^2(Q_{1/2})$ and due to Morrey embedding the velocity field is Hölder continuous in the spatial variables. More precisely, there exists $0 < \gamma < 1$ such that

$$\mathbf{v} \in L^2\big(-1/4, 0; C^\gamma(\overline{B}_{1/2}, \mathbb{R}^3)\big). \tag{5.11}$$

As this is all what is needed for this work, we conclude at this point our discussion of the higher regularity of bounded weak solutions.

Let us instead move on with the derivation of the required energy inequality (5.5). (Remember that in the context of the Navier–Stokes equations with velocity field $\mathbf{v}$, we view the system 5.2 and the energy inequality (5.5) with the notational conventions $\mathbf{A} = \mathbf{v}$, $\mathbf{u} = \partial_i\mathbf{v}$ and $\operatorname{div}(\mathbf{u}\otimes\mathbf{A}) \mapsto 2\operatorname{div}(\mathbf{u}\otimes_s\mathbf{A})$.)

Loosely speaking, this is just a matter of going to the limit $k \to \infty$ in the energy equality above. As a matter of fact, the only terms for which the desired convergence (with the desired limit) does not immediately follow from the previous discussion are those incorporating $\tau_k^i\mathbf{A}\otimes\Delta_k^i\mathbf{v}$. This includes in particular the approximation of the non-harmonic pressure

$p_{1,B}^{k,i}$. But note that, as $k \to 0$, we have

$$\Delta_k^i \mathbf{v} \to \partial_i \mathbf{v} \quad \text{and} \quad \tau_k^i \mathbf{A} \to \mathbf{A}, \tag{5.12}$$

with both convergences being *strongly* in $L^2(B_{1/2} \times (-1/4, 0))$. Thus, we obtain

$$\tau_k^i \mathbf{A} \otimes \Delta_k^i \mathbf{v} \rightharpoonup \mathbf{A} \otimes \partial_i \mathbf{v} \quad \text{in} \quad L^2(B_{1/2} \times (-1/4, 0)), \tag{5.13}$$

and also

$$p_{1,B}^{k,i} \rightharpoonup -2\mathscr{P}_{2,B} \operatorname{div}(\partial_i \mathbf{v} \otimes_s \mathbf{A}) \quad \text{in} \quad L^2(B \times (-1/4, 0)) \tag{5.14}$$

for every $B \subset\subset B_{1/2}$. This really concludes the proof of (5.5) as all the other approximations appearing in the energy equality from above converge strongly in L^2. ♢

The strategy for the proof of 5.2 is pretty much identical to the one for Theorem 4.1. The core of it is again given by an inductive argument, for which a Caccioppoli inequality for the system (5.2) serves as the initial step. More precisely, this Caccioppoli inequality reads as follows.

Proposition 5.5. *Let* $\mathbf{u}$ *be a local suitable weak solution to the system* (5.2) *in the unit parabolic cylinder* Q_1*. Then, the following bound holds for the gradient of the velocity field*

$$\|\nabla \mathbf{u}\|_{L^2(Q_{1/2})}^2 \lesssim (1 \vee \|\mathbf{A}\|_{L^\infty(Q_1)}^2) \|\mathbf{u}\|_{L^2(Q_1)}^2. \tag{5.15}$$

The proportionality constant implicit in this bound is given by an absolute constant.

Proof. First, choose arbitrary real numbers ρ and r, such that $1/2 \le \rho < r \le 1$, and put $\sigma = (r+\rho)/2$. Furthermore, let $\phi \in C_0^\infty(Q)$ be a smooth and in Q_1 compactly supported function with the following properties:

i) $0 \le \phi \le 1$ in Q_1,

ii) $\phi \equiv 0$ in $Q_1 \setminus B_\sigma \times (-\sigma^2, \sigma^2)$,

iii) $\phi \equiv 1$ in Q_ρ, and

iv) $|\phi_t| + |\Delta \phi| + |\nabla \phi|^2 \lesssim (r-\rho)^{-2}$ uniformly in Q_1.

This time, we carry out the local pressure decomposition on the ball B_σ and abbreviate for what follows $\mathbf{v} = \mathbf{v}_{B_\sigma}$ (and so on).

Inserting the test function ϕ^2 into our energy inequality (2.15) yields the bound

$$\operatorname*{ess\,sup}_{t \in (-\sigma^2, 0)} \|\phi(t)\mathbf{v}(t)\|_{L^2(B_\sigma)}^2 + 2\|\phi \nabla \mathbf{v}\|_{L^2(Q_\sigma)}^2 \lesssim J_1 + J_2 + J_3 + J_4, \tag{5.16}$$

where

$$J_1 = \int_{Q_\sigma} |\mathbf{v}|^2 |\phi_t + \Delta\phi| \, dx dt, \quad J_2 = \int_{Q_\sigma} \phi |\mathbf{u}||\mathbf{v}||\mathbf{A}||\nabla\phi| \, dx dt,$$
$$J_3 = \int_{Q_\sigma} \phi^2 |\mathbf{u}||\nabla\mathbf{v}||\mathbf{A}| \, dx dt, \quad J_4 = \int_{Q_\sigma} \phi |p_0||\mathbf{v}||\nabla\phi| \, dx dt.$$

Beginning with J_1, we obtain from the Cattabriga–Solonnikov estimates the bound

$$J_1 \lesssim (r-\rho)^{-2} \|\mathbf{u}\|^2_{L^2(Q_r)}. \tag{5.17}$$

In virtually the same manner, the term J_2 admits the bound

$$J_2 \lesssim (r-\rho)^{-1} \|\mathbf{A}\|_{L^\infty(Q_1)} \|\mathbf{u}\|^2_{L^2(Q_r)}. \tag{5.18}$$

For the third term, we deduce that for every $\delta > 0$

$$\begin{aligned} J_3 &\lesssim \|\mathbf{A}\|_{L^\infty(Q_1)} \|\mathbf{u}\|_{L^2(Q_r)} \|\phi\nabla\mathbf{v}\|_{L^2(Q_\sigma)} \\ &\lesssim \delta^{-1} \|\mathbf{A}\|^2_{L^\infty(Q_1)} \|\mathbf{u}\|^2_{L^2(Q_r)} + \delta \|\phi\nabla\mathbf{v}\|^2_{L^2(Q_\sigma)}. \end{aligned} \tag{5.19}$$

It remains to bound the pressure terms p_1 and p_2. For the latter one, we simply compute for any $\delta > 0$ the bound

$$\begin{aligned} \int_{Q_r} |\mathbf{v}||p_2||\nabla\phi| \, dx dt &\lesssim (r-\rho)^{-1} \|\mathbf{u}\|_{L^2(Q_r)} \|p_2\|_{L^2(Q_\sigma)} \\ &\lesssim \delta^{-1} (r-\rho)^{-2} \|\mathbf{u}\|^2_{L^2(Q_r)} + \delta \|\nabla\mathbf{u}\|^2_{L^2(Q_\sigma)}. \end{aligned} \tag{5.20}$$

For the non-harmonic pressure term, we obtain

$$\begin{aligned} \int_{Q_r} |\mathbf{v}||p_1||\nabla\phi| \, dx dt &\lesssim (r-\rho)^{-1} \|\mathbf{u}\|_{L^2(Q_r)} \|p_1\|_{L^2(Q_r)} \\ &\lesssim (r-\rho)^{-1} \|\mathbf{A}\|_{L^\infty(Q_1)} \|\mathbf{u}\|^2_{L^2(Q_r)}. \end{aligned} \tag{5.21}$$

Finally, note that

$$\|\nabla\mathbf{u}\|^2_{L^2(Q_\rho)} \lesssim \|\phi\nabla\mathbf{v}\|^2_{L^2(Q_\sigma)} + \|\nabla^2 p_h\|^2_{L^2(Q_\rho)}, \tag{5.22}$$

and by the classical Caccioppoli inequality for harmonic functions

$$\|\nabla^2 p_h\|^2_{L^2(Q_\rho)} \lesssim (r-\rho)^{-2} \|\mathbf{u}\|^2_{L^2(Q_\sigma)}. \tag{5.23}$$

Choosing $\delta > 0$ sufficiently small in (5.19) and (5.20), we infer from our estimates (5.16) to (5.23) that

$$\|\nabla\mathbf{u}\|^2_{L^2(Q_\rho)} \lesssim (r-\rho)^{-2} (1 \vee \|\mathbf{A}\|^2_{L^\infty(Q_1)}) \|\mathbf{u}\|^2_{L^2(Q_1)} + \frac{1}{2} \|\nabla\mathbf{u}\|^2_{L^2(Q_r)}.$$

Thus, we are precisely in the situation of Proposition A.1, and the desired bound in (5.15) follows immediately from (A.2). ■

The above Caccioppoli inequality for local suitable weak solutions was given in terms of the unit parabolic cylinder. How about general parabolic cylinders $Q_R(x_0,t_0)=(t_0-R^2,t_0)\times B_R(x_0)$? In this case, note first that (local suitable weak) solutions to the system given in (5.2) obey the following scaling symmetry:

$$\begin{aligned}\mathbf{u}(t,x)&\mapsto \lambda\mathbf{u}\big(\lambda(x-x_0),\lambda^2(t-t_0)\big),\\ p(t,x)&\mapsto \lambda^2 p\big(\lambda(x-x_0),\lambda^2(t-t_0)\big),\\ \mathbf{A}(t,x)&\mapsto \lambda\mathbf{A}\big(\lambda(x-x_0),\lambda^2(t-t_0)\big).\end{aligned}$$

In particular, solutions defined in $Q_R(x_0,t_0)$ can be rescaled to solutions in parabolic cylinders with top-centre point at $(0,0)\in\mathbb{R}^3\times\mathbb{R}$.

Secondly, let us introduce the following scale-invariant quantities:

$$\begin{aligned}A(R)&=R^{-2}\|\mathbf{A}\|^2_{L^\infty(Q_R(x_0,t_0))}, &\quad B(R)&=R^{-1}\|\nabla\mathbf{u}\|^2_{L^2(Q_R(x_0,t_0))}\\ C(R)&=R^{-1}\|\mathbf{u}\|^2_{L^{2,\infty}(Q_R(x_0,t_0))}, &\quad D(R)&=R^{-3}\|\mathbf{u}\|^2_{L^2(Q_R(x_0,t_0))}.\end{aligned}\tag{5.24}$$

(Space-time translations do not play any particular role, so that we omit again the location of the top-centre point in the notation.) With this notation at hand, our Caccioppoli inequality from (5.15) reads as follows in scale-invariant fashion:

$$B(R/2)\lesssim(1\vee A(R))D(R).\tag{5.25}$$

Being equipped with this Caccioppoli inequality, we are now in position to prove the pendant of Proposition 4.2 for the system (5.2). To this end, let us first recall the following notation:

$$r_n=2^{-n},\quad I^n=(-r_n^2,0),\quad B^n=B_{r_n}(0),\quad Q^n=B^n\times I^n,\quad n\geq 1.$$

Furthermore, we fix

$$B=B^2,\quad p_h=p_{h,B},\quad p_1=p_{1,B},\quad p_2=p_{2,B},\quad \mathbf{v}=\mathbf{u}+\nabla p_h.$$

Proposition 5.6. *Let $\mathbf{u}$ be a local suitable weak solution to the system* (5.2) *in the unit parabolic cylinder Q_1. There exists an integer $n_0\geq 2$, such that for all $n\geq n_0$, the following bound holds*

$$r_n^{-3}\operatorname*{ess\,sup}_{I^n}\int_{B^n}|\mathbf{u}|^2dx+r_n^{-3}\int_{Q^n}|\nabla\mathbf{u}|^2dxdt\lesssim 2^{3n_0}(1\vee A(1/2))C(1/2).\tag{5.26}$$

Furthermore, the integer n_0 just depends on the quantity $1\vee A(1)$. The constant hidden in the bound (5.26) *is given by the maximum of 1 and the hidden constant in our Caccioppoli inequality* (5.15)*. (For the definition of the various scale-invariant quantities, see* (5.24)*.)*

Proof. We argue by induction on $n \geq n_0$ and start with the

Induction basis ($n = n_0$)**:** Due to our Caccioppoli inequality in (5.25), there is nothing to prove here, so we immediately move on to the

Induction step ($n-1 \mapsto n$)**:** So let $n \geq n_0+1$. At the end of the induction step, we will observe how to choose $n_0 \in \mathbb{N}$. For obvious reasons, we again make use of a suitably regularized version for the fundamental solution to the backward heat equation. This time, choose as a regularizer a cut-off function χ which is smooth at least on $\mathbb{R}^3 \times (-\infty, 0]$, and which has in addition the following properties: $0 \leq \chi \leq 1$, $\chi \equiv 1$ in Q^{n_0+1} and $\chi \equiv 0$ outside of $Q_{\widetilde{r}_{n_0}}$, where $\widetilde{r}_{n_0} = 1/2(r_{n_0+1} + r_{n_0})$. Put

$$\phi_n := \chi \psi_n \geq 0,$$

where ψ_n denotes the fundamental solution of the backward heat equation with singularity at $(0, r_n^2)$.

Now, we insert ϕ_n in our generalized energy inequality (5.5). We infer from the bound in (4.6) that

$$r_n^{-3} \|\mathbf{v}\|^2_{L^{2,\infty}(Q^n)} + 2 r_n^{-3} \|\nabla \mathbf{v}\|^2_{L^2(Q^n)} \lesssim J_1 + J_2 + J_3 + J_4 + J_5, \tag{5.27}$$

where

$$J_1 = \int_{Q^{n_0}} |\mathbf{v}|^2 \left|\partial_t \phi_n + \Delta \phi_n\right| \, dx dt, \quad J_2 = \int_{Q^{n_0}} |\mathbf{u}||\mathbf{v}||\mathbf{A}||\nabla \phi_n| \, dx dt,$$

$$J_3 = \int_{I^{n_0}} \left| \int_{B^{n_0}} p_1 \mathbf{v} \cdot \nabla \phi_n dx \right| dt, \qquad J_4 = \int_{I^{n_0}} \left| \int_{B^{n_0}} p_2 \mathbf{v} \cdot \nabla \phi_n dx \right| dt,$$

$$J_5 = \int_{Q^{n_0}} |\mathbf{u}||\nabla \mathbf{v}||\mathbf{A}| \phi_n \, dx dt.$$

Before we proceed with all of these individual terms, let us make the following elementary observation from the induction hypothesis

$$\begin{aligned} r_n^{-3} \operatorname*{ess\,sup}_{I^n} \int_{B^n} |\mathbf{u}|^2 dx &\leq 2^3 r_{n-1}^{-3} \operatorname*{ess\,sup}_{I^{n-1}} \int_{B^{n-1}} |\mathbf{u}|^2 dx \\ &\lesssim 2^{3n_0} (1 \vee A(1/2)) C(1/2). \end{aligned} \tag{5.28}$$

As another preparation for what follows, let us take a closer look at the (harmonic) pressure term p_h. For this, let $k \in \{n_0, \ldots, n\}$. On one side, we can exploit the mean value inequality for harmonic functions to get

$$\begin{aligned} r_k^{-5} \int_{Q^k} |\nabla p_h|^2 dx dt &\lesssim r_k^{-5} \int_{Q^k} \left| \operatorname*{ess\,sup}_{I^{n_0}} \|\nabla p_h\|_{L^1(B^2)} \right|^2 dx dt \\ &\lesssim \operatorname*{ess\,sup}_{I^2} \|\nabla p_h\|^2_{L^2(B^2)}. \end{aligned} \tag{5.29}$$

On the other side, using in addition the classical Caccioppoli inequality for harmonic functions, we derive the following bound ($\widetilde{r}_{n_0} \le 3/16$)

$$\begin{aligned} r_k^{-5}\int_{Q^k}\phi_n^2|\nabla^2 p_h|^2\,dxdt &\lesssim r_k^{-5}\int_{Q^k}\left|\operatorname*{ess\,sup}_{I^{n_0}}\|\nabla^2 p_h\|_{L^1(B_{3/16})}\right|^2 dxdt \\ &\lesssim \operatorname*{ess\,sup}_{I^2}\|\nabla^2 p_h\|^2_{L^2(B_{3/16})} \\ &\lesssim \operatorname*{ess\,sup}_{I^2}\|\nabla p_h\|^2_{L^2(B^2)}. \end{aligned} \tag{5.30}$$

Now, let us begin to estimate term by term in (5.27). Due to the bound in (4.5), the term J_1 can be estimated immediately as follows

$$J_1 \lesssim \|\mathbf{u}\|^2_{L^2(Q^2)} \lesssim D(1/2). \tag{5.31}$$

For the second term, we make use of (4.7), (5.28), (5.29) and the induction hypothesis to obtain the bound

$$\begin{aligned} J_2 &\lesssim \|\mathbf{A}\|_{L^\infty(Q^{n_0})}\sum_{k=n_0}^{n} r_k\left(r_k^{-5}\int_{Q^k}|\mathbf{u}|^2dxdt\right)^{\frac12}\left(r_k^{-5}\int_{Q^k}|\mathbf{v}|^2dxdt\right)^{\frac12} \\ &\lesssim (1\vee A(1))2^{-n_0}2^{3n_0}(1\vee A(1/2))C(1/2). \end{aligned} \tag{5.32}$$

Furthermore, we can estimate

$$\begin{aligned} J_5 \lesssim\ & \|\mathbf{A}\|_{L^\infty(Q^{n_0})}\sum_{k=n_0}^{n-1} r_k^2\left(r_k^{-5}\int_{Q^k}|\mathbf{u}|^2dxdt\right)^{\frac12}\left(r_k^{-5}\int_{Q^k}|\nabla\mathbf{v}|^2dxdt\right)^{\frac12} + \\ &+\|\mathbf{A}\|_{L^\infty(Q^{n_0})} r_n^2\left(r_n^{-5}\int_{Q^n}|\mathbf{u}|^2dxdt\right)^{\frac12}\left(r_n^{-5}\int_{Q^n}|\nabla\mathbf{v}|^2dxdt\right)^{\frac12}. \end{aligned}$$

Hence, the induction hypothesis, Young's inequality and (5.28) as well as (5.30) yield an appropriate bound for J_5:

$$J_5 \lesssim (1\vee A(1))(2^{-n_0}2^{3n_0}+2^{\frac32 n_0})(1\vee A(1/2))C(1/2)+2^{-\frac{n_0}{2}}r_n^{-3}\int_{Q^n}|\nabla\mathbf{v}|^2dxdt.$$

In fact, the proportionality constant in front of the second term is just a multiple of the absolute constant $c>0$ from (4.6). In other words, choosing n_0 appropriately enables us to absorb this term into the left hand side of (5.27). Hence, all what is essential for us is contained in the bound

$$J_5 \lesssim (1\vee A(1))(2^{-n_0}2^{3n_0}+2^{\frac32 n_0})(1\vee A(1/2))C(1/2). \tag{5.33}$$

This leaves us with the pressure terms. To this end, let us begin again with J_4 as the pressure term p_2 is harmonic. As in the proof of Proposition 4.2, we obtain the bound

$$J_4 \lesssim \sum_{k=n_0+1}^{n} r_k^{-4}\int_{Q^k}|\mathbf{v}||p_2-[p_2]_{B^k}|\,dxdt+\int_{Q^2}|\mathbf{v}||p_2|\,dxdt =: J_4'+J_4''. \tag{5.34}$$

We can bound the latter term by means of our Caccioppoli inequality (5.25) as follows

$$J_4'' \lesssim \|\mathbf{u}\|^2_{L^2(Q^2)} + \|\nabla \mathbf{u}\|^2_{L^2(Q^2)} \lesssim (1 \vee A(1/2))C(1/2). \tag{5.35}$$

Next, we estimate

$$J_4' \lesssim \sum_{k=n_0+1}^{n} r_k \left(r_k^{-5} \int_{Q^k} |\mathbf{v}|^2 dxdt \right)^{\frac{1}{2}} \left(r_k^{-5} \int_{Q^k} |p_2 - [p_2]_{B^k}|^2 dxdt \right)^{\frac{1}{2}}.$$

As p_2 is harmonic, we obtain for all $k \in \{n_0+1, \ldots, n\}$

$$\begin{aligned} \left(r_k^{-5} \int_{Q^k} |p_2 - [p_2]_{B^k}|^2 dxdt \right)^{\frac{1}{2}} &\lesssim \left(r_k^{-5} \int_{I^k} \left(\frac{r_k}{2^2}\right)^5 \int_{B^2} |p_2 - [p_2]_{B^2}|^2 dxdt \right)^{\frac{1}{2}} \\ &\lesssim \|\nabla \mathbf{u}\|_{L^2(Q^2)}, \end{aligned}$$

and thus also

$$J_4 \lesssim (1 + 2^{-n_0}) 2^{\frac{3}{2} n_0} (1 \vee A(1/2))C(1/2). \tag{5.36}$$

For the non-harmonic pressure field, we again start with the bound

$$J_3 \lesssim \sum_{k=n_0+1}^{n} r_k^{-4} \int_{Q^k} |\mathbf{v}||p_1 - [p_1]_{B^k}| dxdt + \int_{Q^2} |\mathbf{v}||p_1| dxdt =: J_3' + J_3''.$$

An elementary computation shows that

$$J_3'' \lesssim A(1/2)D(1/2).$$

The other term is treated similarly as in the proof of Proposition 4.2, i.e. the following inequality is satisfied for all choices of $r_n \le \rho < r \le r_{n_0}$:

$$\begin{aligned} &\int_{Q_\rho} \left| p_1 - [p_1]_{B_\rho} \right|^2 dxdt \lesssim \\ &\qquad \lesssim 2^{3n_0} (1 \vee A(1/2))C(1/2) \rho^4 + \left(\frac{\rho}{r}\right)^4 \int_{Q_r} \left| p_1 - [p_1]_{B_r} \right|^2 dxdt. \end{aligned}$$

In particular, this entails for all $k \in \{n_0+1, \ldots, n\}$ the bound

$$\begin{aligned} r_k^{-4} \int_{Q^k} \left| p_1 - [p_1]_{B^k} \right|^2 dxdt &\lesssim 2^{3n_0} (1 \vee A(1/2))C(1/2) + \int_{Q^2} |p_1|^2 dxdt \\ &\lesssim 2^{3n_0} (1 \vee A(1/2))C(1/2) + \|\mathbf{u}\|^2_{L^2(Q^2)} \\ &\lesssim 2^{3n_0} (1 \vee A(1/2))C(1/2). \end{aligned}$$

Eventually, we are in position to derive a bound for J_3' as follows

$$\begin{aligned} J_3' &\lesssim \sum_{k=n_0+1}^{n} r_k^{\frac{1}{2}} \left(r_k^{-5} \int_{Q^k} |\mathbf{v}|^2 dxdt \right)^{\frac{1}{2}} \left(r_k^{-4} \int_{Q^k} \left| p_1 - [p_1]_{B^k} \right|^2 dxdt \right)^{\frac{1}{2}} \\ &\lesssim 2^{-\frac{n_0}{2}} 2^{3n_0} (1 \vee A(1/2))C(1/2). \end{aligned}$$

Therefore, a bound for the term involving the non-harmonic pressure p_1 is then given by

$$J_3 \lesssim \left(1 + 2^{-\frac{n_0}{2}} 2^{3n_0}\right)(1 \vee A(1/2))C(1/2). \tag{5.37}$$

Now, from our bounds in (5.31), (5.32), (5.33), (5.36) and (5.37), we can deduce that

$$\begin{aligned} &r_n^{-3}\|\mathbf{v}\|^2_{L^{2,\infty}(Q^n)} + r_n^{-3}\|\nabla\mathbf{v}\|^2_{L^2(Q^n)} \\ &\qquad \lesssim (1 \vee A(1))2^{-\frac{n_0}{2}} 2^{3n_0}(1 \vee A(1/2))C(1/2). \end{aligned} \tag{5.38}$$

But remind that

$$\|\phi_n^{1/2}\nabla p_h\|^2_{L^{2,\infty}(Q^n)} \vee \|\phi_n^{1/2}\nabla^2 p_h\|^2_{L^2(Q^n)} \lesssim C(1/2)$$

due to our computation in (5.29) and (5.30). Thus, the bound in (5.38) also holds when we replace $\mathbf{v}$ by $\mathbf{u}$. As the proportionality constant implicit in this bound does not depend on any of our parameters, all what is left to do is to choose n_0 sufficiently large in order to conclude the proof. ■

By a simple scaling argument and interpolation inequality (A.10), we obtain from the result above our Theorem 5.2.

Proof. Consider some arbitrary space-time point $(y_0, s_0) \in Q_{1/2}(0,0)$ and let

$$\begin{aligned} \widetilde{\mathbf{u}}(x,t) &= 2^{-1}\mathbf{u}\left(y_0 + 2^{-1}x, s_0 + 2^{-2}t\right), \\ \widetilde{\mathbf{A}}(x,t) &= 2^{-1}\mathbf{A}\left(y_0 + 2^{-1}x, s_0 + 2^{-2}t\right) \end{aligned}$$

for $(x,t) \in Q_1$. Note that $\widetilde{\mathbf{u}}$ is a local suitable weak solution for (1.1) in Q_1 with respect to $\widetilde{\mathbf{A}}$ and

$$\|\widetilde{\mathbf{u}}\|^3_{L^{2,\infty}(Q_1)} = 2^{\frac{3}{2}}\|\mathbf{u}\|^3_{L^{2,\infty}(Q_{1/2}(y_0,s_0))} \lesssim \|\mathbf{u}\|^3_{L^{2,\infty}(Q_1)}.$$

In addition, by interpolation inequality (A.10) and from our bound in (5.26) we also obtain for every $n \geq n_0$ that

$$\begin{aligned} r_n^{-5}\|\widetilde{\mathbf{u}}\|^3_{L^3(Q^n)} &\lesssim \left(r_n^{-3}\|\widetilde{\mathbf{u}}\|^2_{L^{2,\infty}(Q^n)} + r_n^{-3}\|\nabla\widetilde{\mathbf{u}}\|^2_{L^2(Q^n)}\right)^{\frac{3}{2}} \\ &\lesssim 2^{\frac{9}{2}n_0}\left(1 \vee \|\mathbf{A}\|_{L^\infty(Q_1)}\right)^{\frac{3}{2}}\|\widetilde{\mathbf{u}}\|^3_{L^{2,\infty}(Q_1)}. \end{aligned}$$

Here, it is crucial to note that n_0 can be chosen independently of the choice of $(y_0, s_0) \in Q_{1/2}(0,0)$. As a matter of fact, this n_0 just depends on the quantity $1 \vee \|\mathbf{A}\|_{L^\infty(Q_1)}$, as the proof of Proposition 5.6 reveals. Thus, a simple change of variables shows that the bound

$$r_{n+1}^{-5}\int_{Q^{n+1}(y_0,s_0)} |\mathbf{u}(y,s)|^3\,dy\,ds \lesssim 2^{\frac{9}{2}n_0}\left(1 \vee \|\mathbf{A}\|_{L^\infty(Q_1)}\right)^{\frac{3}{2}}\|\mathbf{u}\|^3_{L^{2,\infty}(Q_1)} \tag{5.39}$$

holds uniformly over all $n \geq n_0$ and $(y_0, s_0) \in Q_{1/2}(0,0)$. But as almost every $(y_0, s_0) \in Q_{1/2}(0,0)$ is a Lebesgue point of $\mathbf{u}$, the assertion follows directly from this last bound. ■

5.2 Application to spatial derivatives

As an application of our results of the previous subsection, we want to establish the announced local L^∞-bound for the spatial derivatives of a local suitable weak solution of the Navier–Stokes equations (i.e. Theorem 5.1). In the following lemma, we summarize the situation which shall underlie this whole section. For the proof, we refer to Example 5.4.

Lemma 5.7. *Let* $\mathbf{v}$ *be a local suitable weak solution of the Navier–Stokes equations in the unit parabolic cylinder* Q_1 *with* $\|\mathbf{v}\|^2_{L^{2,\infty}(Q_1)} \leq \varepsilon^2$. *Here,* $\varepsilon > 0$ *is the absolute constant from Theorem 4.1. In this situation, all assertions in Example 5.4 hold.*

In particular, each spatial derivative of $\mathbf{v}$ *is a local suitable weak solution of* (5.2) *in the half parabolic cylinder* $Q_{1/2}$. *(Of course, this is meant in the sense that the convection term in* (5.2) *is replaced by* $2\operatorname{div}(\mathbf{u} \otimes_s \mathbf{A})$.*)*

For future reference, let us restate the corresponding generalized energy inequality which holds for every ball $B \subset\subset B_{1/2}$, *every non-negative test function* $\phi \in C_c^\infty(B \times (-1/4, 0))$ *and a.e.* $t \in (-1/4, 0)$:

$$\int_B |\mathbf{w}_B(t)|^2 \phi(t)\,dx + 2\int_{-\frac{1}{4}}^t \int_B |\nabla \mathbf{w}_B|^2 \phi\,dx\,ds \leq$$

$$\leq \int_{-\frac{1}{4}}^t \int_B |\mathbf{w}_B|^2 (\partial_t \phi + \Delta\phi)\,dx\,ds + 2\int_{-\frac{1}{4}}^t \int_B (q_{1,B} + q_{2,B})\mathbf{w}_B \cdot \nabla\phi\,dx\,ds \tag{5.40}$$

$$+ 2\int_{-\frac{1}{4}}^t \int_B (\partial_i \mathbf{v} \otimes_s \mathbf{v}) : (\phi \nabla \mathbf{w}_B + \mathbf{w}_B \otimes \nabla\phi)\,dx\,ds,$$

where $\mathbf{w}_B = \partial_i \mathbf{v} + \nabla q_{h,B}$ *and*

$$q_{h,B} = -\mathscr{P}_{2,B}\partial_i \mathbf{v}, \quad q_{1,B} = -\mathscr{P}_{2,B}\big(\operatorname{div}(\partial_i \mathbf{v} \otimes_s \mathbf{v})\big), \quad q_{2,B} = \mathscr{P}_{2,B}\Delta(\partial_i \mathbf{v}).$$

In order to establish the desired bound for the derivatives, we will first prove the following lemma which constitutes the pendant to the Caccioppoli inequality (3.3). It also sharpens in a certain sense the Caccioppoli inequality for the system (5.2).

Lemma 5.8. *Let* $\mathbf{v}$ *be a local suitable weak solution to the Navier–Stokes equations in the unit parabolic cylinder* Q_1 *such that* $\|\mathbf{v}\|^2_{L^{2,\infty}(Q_1)} \leq \varepsilon^2$. *Here,* $\varepsilon > 0$ *is the absolute constant from Theorem 4.1. Then, the following bound holds for every spatial derivative of* $\mathbf{v}$

$$\|\partial_i \mathbf{v}\|^2_{L^{2,\infty}(Q^4)} + \|\nabla \partial_i \mathbf{v}\|^2_{L^2(Q^4)} \lesssim \|\mathbf{v}\|^2_{L^{2,\infty}(Q_1)}. \tag{5.41}$$

The proportionality constant implicit in this bound is an absolute constant.

Remark 5.9. Due to our Caccioppoli inequality for the system (5.2) and the Caccioppoli inequality in (3.3), there is no difficulty in order to obtain the desired bound for $\nabla\partial_i \mathbf{v}$. But for the first term, some specific work is needed here. $\diamondsuit$

Proof. Choose a smooth cut-off function ϕ such that $0 \le \phi \le 1$, $\phi \equiv 1$ in Q^4 and $\operatorname{supp}\phi \subset Q^3$. Next, insert ϕ^2 into the generalized energy inequality (5.40). This yields the bound

$$\|\phi\mathbf{w}\|^2_{L^{2,\infty}(Q^3)} + 2\|\phi\nabla\mathbf{w}\|^2_{L^2(Q^3)} \lesssim J_1 + J_2 + J_3 + J_4 + J_5, \tag{5.42}$$

where

$$J_1 = \int_{Q^3} |\mathbf{w}|^2 \left|\partial_t\phi + \Delta\phi\right| \, dx\,dt, \qquad J_2 = \int_{Q^3} \phi|\partial_i\mathbf{v}||\mathbf{w}||\mathbf{v}||\nabla\phi| \, dx\,dt,$$

$$J_3 = \int_{Q^3} \phi^2|\partial_i\mathbf{v}||\nabla\mathbf{w}||\mathbf{v}| \, dx\,dt, \qquad J_4 = \int_{Q^3} \phi|q_2||\mathbf{w}||\nabla\phi| \, dx\,dt,$$

$$J_5 = \int_{Q^3} \phi|q_1||\mathbf{w}||\nabla\phi| \, dx\,dt.$$

On multiple occasions, we will encounter the term $\|\mathbf{v}\|_{L^\infty(Q^3)}$. Due to Theorem 4.1 and the fact that $\varepsilon \le 1$, we can estimate this term as follows

$$\|\mathbf{v}\|_{L^\infty(Q^3)} \le \sqrt{K_0}\|\mathbf{v}\|_{L^{2,\infty}(Q_1)} \lesssim 1.$$

In other words, we will forget about it in the following.

Now, let us estimate each term in (5.42). For the first term, we make use of the Cattabriga–Solonnikov estimate $\|\nabla q_h\|^2_{L^2(Q^2)} \lesssim \|\partial_i\mathbf{v}\|^2_{L^2(Q^2)}$ and the Caccioppoli inequality in (3.3), i.e.

$$J_1 \lesssim \|\partial_i\mathbf{v}\|^2_{L^2(Q^2)} + \|\nabla q_h\|^2_{L^2(Q^2)} \lesssim \|\mathbf{v}\|^2_{L^{2,\infty}(Q_1)}. \tag{5.43}$$

The second term can be treated along the same lines. Hence, we obtain

$$J_2 \lesssim \|\mathbf{v}\|_{L^\infty(Q^3)}\|\partial_i\mathbf{v}\|_{L^2(Q^2)}\|\mathbf{w}\|_{L^2(Q^2)} \lesssim \|\mathbf{v}\|^2_{L^{2,\infty}(Q_1)}. \tag{5.44}$$

The third term involves the gradient of $\mathbf{w}$, i.e. we have to find suitable bounds for $\nabla^2 q_h$ and $\nabla\partial_i\mathbf{v}$. For the latter one, we simply employ our Caccioppoli inequality from (5.15), and for the other one, we exploit the classical Caccioppoli inequality for harmonic functions. Therefore, we can compute

$$J_3 \lesssim \|\partial_i\mathbf{v}\|_{L^2(Q^2)}(\|\nabla\partial_i\mathbf{v}\|_{L^2(Q^2)} + \|\nabla q_h\|_{L^2(Q^2)}) \lesssim \|\mathbf{v}\|^2_{L^{2,\infty}(Q_1)}. \tag{5.45}$$

For the term involving q_2, note first that $\|q_2\|^2_{L^2(Q^2)} \lesssim \|\nabla\partial_i\mathbf{v}\|^2_{L^2(Q^2)}$ due to (2.2). So as above, we deduce

$$J_4 \lesssim \|q_2\|_{L^2(Q^2)}\|\mathbf{w}\|_{L^2(Q^2)} \lesssim \|\mathbf{v}\|^2_{L^{2,\infty}(Q_1)}. \tag{5.46}$$

Finally, we remark that $\|q_1\|^2_{L^2(Q^2)} \lesssim \|\partial_i \mathbf{v} \otimes_s \mathbf{v}\|^2_{L^2(Q^2)}$ again by (2.2). But as $\mathbf{v}$ is bounded in $Q_{1/2}$, we infer

$$J_5 \lesssim \|q_2\|_{L^2(Q^2)} \|\mathbf{w}\|_{L^2(Q^2)} \lesssim \|\partial_i \mathbf{v}\|_{L^2(Q^2)} \|\mathbf{w}\|_{L^2(Q^2)} \lesssim \|\mathbf{v}\|^2_{L^{2,\infty}(Q_1)}. \tag{5.47}$$

All these estimates together imply that

$$\|\phi \mathbf{w}\|^2_{L^{2,\infty}(Q^4)} \lesssim \|\phi \mathbf{w}\|^2_{L^{2,\infty}(Q^3)} \lesssim \|\mathbf{v}\|^2_{L^{2,\infty}(Q_1)}. \tag{5.48}$$

In the final step, we want to replace $\mathbf{w}$ with $\mathbf{v}$ on the left hand side of this last inequality. To this end, we begin with the following elementary observation

$$\|\mathbf{v}\|^2_{L^{2,\infty}(Q^4)} \lesssim \|\phi \mathbf{w}\|^2_{L^{2,\infty}(Q^3)} + \|\nabla q_h\|^2_{L^{2,\infty}(Q^3)}.$$

The problem with the last term is that the Cattabriga–Solonnikov estimate $\|\nabla q_h\|^2_{L^{2,\infty}(Q_2)} \lesssim \|\partial_i \mathbf{v}\|^2_{L^{2,\infty}(Q^2)}$ is of no help for us, as the quantity $\|\partial_i \mathbf{v}\|^2_{L^{2,\infty}(Q^2)}$ admits no bound in terms of $\|\mathbf{v}\|^2_{L^{2,\infty}(Q_1)}$. So, what can we do instead?

First recall that q_h is harmonic, i.e. the classical Caccioppoli inequality for harmonic functions guarantees

$$\|\nabla q_h\|^2_{L^{2,\infty}(Q^3)} \lesssim \|q_h\|^2_{L^{2,\infty}(Q^2)}.$$

On the other side, the uniqueness statement in Proposition 2.1 entails that

$$q_h = \partial_i \mathscr{P}_{2,B^2} \mathbf{v} - \left[\partial_i \mathscr{P}_{2,B^2} \mathbf{v}\right]_{B^2}.$$

(The correction term appears, because $q_h \in L^2(-1/16, 0; L^2_0(B^2))$.) Hence, we obtain the bound

$$\|\nabla q_h\|^2_{L^{2,\infty}(Q^3)} \lesssim \|\nabla \mathscr{P}_{2,B^2} \mathbf{v}\|^2_{L^{2,\infty}(Q^2)} \lesssim \|\mathbf{v}\|^2_{L^{2,\infty}(Q_1)}. \tag{5.49}$$

The assertion eventually follows from this last observation, the bound in (5.48) and the remark preceding this proof. ■

We are now in a position to prove Theorem 5.1. As a matter of fact, we can choose $\varepsilon_1 \sim \varepsilon$. Therefore, if $\|\mathbf{v}\|^2_{L^{2,\infty}(Q_1)} \leq \varepsilon_1^2$ then the last lemma together with (a rescaled version of) Theorem 5.2 yield the bound

$$\|\partial_i \mathbf{v}\|^2_{L^\infty(Q^5)} \lesssim \|\mathbf{v}\|^2_{L^{2,\infty}(Q_1)}.$$

Since $\varepsilon \sim \frac{1}{K}$ in Theorem 4.1, the proportionality constant implicit in this bound is an absolute constant. Similar to the proof of Theorem 5.2, an appropriate rescaling argument shows on one side, how to choose the (absolute) constant K_1 and on the other side that we indeed obtain from this the desired bound

$$\|\partial_i \mathbf{v}\|^2_{L^\infty(Q_{1/2})} \leq K_1 \|\mathbf{v}\|^2_{L^{2,\infty}(Q_1)}. \tag{5.50}$$

A Appendix

For reader's convenience and for reference purposes, we collect in this short appendix the most essential techniques and results which are used throughout the text.

Iteration procedures We start with two important and very useful iteration schemes. Even though both of them are well known results, we want to present their proofs here since the proofs are short and both iteration schemes are fundamental tools. For instance, they are key to the regularity theory of elliptic equations whenever one wants to avoid potential theory (cf. [9]). This approach to regularity theory is exposed, for example, in the textbooks of Giaquinta [8, 9] whereat the ideas itself already date back to the works of Morrey [16] and Campanato [2].

Proposition A.1. *Let* $\Psi\colon [a,b] \to [0,\infty)$ *be a bounded function,* $0 \le a < b$*. Furthermore, assume that there are non-negative constants* A, B, γ, θ *with* $\theta < 1$ *such that*

$$\Psi(\rho) \le A + B(r-\rho)^{-\gamma} + \theta\Psi(r) \tag{A.1}$$

holds for all $a \le \rho < r \le b$*. Then there exists a constant* $c = c(\gamma,\theta)$ *such that*

$$\Psi(a) \le c\left\{A + B(b-a)^{-\gamma}\right\}. \tag{A.2}$$

Proof. We proceed as in [8]. Let $0 < \alpha < 1$ and consider the sequence $(r_n)_{n\in\mathbb{N}_0}$, which we define recursively by

$$r_0 = a, \quad r_{k+1} = r_k + (1-\alpha)\alpha^k(b-a)$$

for $k \in \mathbb{N}_0$. Iterating inequality (A.1) yields for all $n \in \mathbb{N}$

$$\begin{aligned}\Psi(a) &\le A\sum_{k=0}^{n-1}\theta^k + B\sum_{k=0}^{n-1}\theta^k(r_{k+1}-r_k)^{-\gamma} + \theta^n\Psi(r_n)\\ &\le \frac{A}{1-\theta} + \frac{B}{(1-\alpha)^\gamma}(b-a)^{-\gamma}\sum_{k=0}^{n-1}(\theta\alpha^{-\gamma})^k + M\theta^n,\end{aligned}$$

where M is an upper bound for Ψ on $[a,b]$. A judicious choice of α enables us to take the limit $n \to \infty$, i.e.

$$\Psi(a) \le \frac{A}{1-\theta} + \frac{B}{(1-\alpha)^\gamma(1-\theta\alpha^{-\gamma})}(b-a)^{-\gamma}, \tag{A.3}$$

which finishes the proof. ■

Proposition A.2. *Let* $\Psi\colon [a,b] \to [0,\infty)$ *be a function,* $0 < a < b \le 1$*, such that for all* $a \le \rho < r \le b$ *and some absolute constant* $M > 0$ *the inequality*

$\Psi(\rho) \le M\Psi(r)$ holds. Furthermore, assume that there are non-negative constants A, B, α, β with $0 < \alpha < \beta$ and $B > 1$, such that the inequality

$$\Psi(\rho) \le A r^{\alpha} + B\left(\frac{\rho}{r}\right)^{\beta} \Psi(r) \tag{A.4}$$

holds for all $a \le \rho < r \le b$. Then there exists a constant $c = c(M, B, \alpha, \beta)$ such that

$$\Psi(\rho) \le c\left\{A\rho^{\alpha} + \left(\frac{\rho}{r}\right)^{\alpha} \Psi(r)\right\} \tag{A.5}$$

is true for all $a \le \rho < r \le b$.

Proof. We follow the argumentation in [28]. Put $\theta = (2B)^{1/(\beta-\alpha)}$ and choose (the unique) $k \in \mathbb{N}_0$ such that $\theta^k < r/\rho \le \theta^{k+1}$. If $k = 0$ we immediately obtain

$$\Psi(\rho) \le M\Psi(r) \le M\theta^{\alpha}\left(\frac{\rho}{r}\right)^{\alpha}\Psi(r) = M(2B)^{\frac{\alpha}{\beta-\alpha}}\left(\frac{\rho}{r}\right)^{\alpha}\Psi(r).$$

If $k \ge 1$ iterating inequality (A.4) yields

$$\begin{aligned}\Psi(\rho) &\le M\Psi(r\theta^{-k}) \le M\left\{r^{\alpha}\theta^{-\alpha(k-1)}A + B\theta^{-\beta}\Psi\left(r\theta^{-(k-1)}\right)\right\}\\ &\le M\left\{r^{\alpha}\theta^{-\alpha(k-1)}A\sum_{j=0}^{k-1}\left(\theta^{\alpha-\beta}B\right)^{j} + B^{k}\theta^{-k\beta}\Psi(r)\right\}.\end{aligned}$$

Since $\theta^{\alpha-\beta}B = 1/2$ and $\theta^{-\alpha(k+1)}r^{\alpha} \le \rho^{\alpha}$, we obtain

$$\begin{aligned}\Psi(\rho) &\le M\theta^{-\alpha(k+1)}\left\{A\theta^{2\alpha}r^{\alpha} + \left(\theta^{\alpha-\beta}B\right)^{k}\theta^{\alpha}\Psi(r)\right\}\\ &\le M(2B)^{\frac{2\alpha}{\beta-\alpha}}A\rho^{\alpha} + M(2B)^{\frac{\alpha}{\beta-\alpha}}\left(\frac{\rho}{r}\right)^{\alpha}\Psi(r).\end{aligned}$$

Hence, the claim follows with $c = c(M, B, \alpha, \beta) = M(2B)^{2\alpha/(\beta-\alpha)}$. ■

Estimates for harmonic functions Next, we state some important estimates for harmonic functions. To this end, let $V \subset \mathbb{R}^d$ be a domain and $\mathbf{v} \in H^1(V, \mathbb{R}^m)$ a weak solution to $\Delta\mathbf{v} = 0$ in V, i.e. Laplace's equation shall hold in the sense of distributions. By a standard argument one obtains that for all $x_0 \in V$ and $\rho < r < \mathrm{dist}(x_0, \partial V)$ the following inequality holds

$$\int_{B_\rho(x_0)} |\nabla\mathbf{v}|^2\,\mathrm{d}x \le \frac{c}{(r-\rho)^2}\int_{B_r(x_0)} |\mathbf{v}|^2\,\mathrm{d}x, \tag{A.6}$$

where $c > 0$ is an absolute constant.

Proof. Being a weak solution to Laplace's equation precisely means that for all smooth test functions $\boldsymbol{\varphi} \in C_c^\infty(V, \mathbb{R}^m)$ the following equation holds

$$\int_V \nabla \mathbf{v} : \nabla \boldsymbol{\varphi} \, \mathrm{d}x = 0.$$

Now, choose a cut-off function $\phi \in C_c^\infty(V)$ with support in $B_r(x_0)$, such that $0 \le \phi \le 1$, $\phi \equiv 1$ in $B_\rho(x_0)$ and $|\nabla \phi| \le c/(r-\rho)$, where $c > 0$ is an absolute constant. Note that $\boldsymbol{\varphi} = \phi^2 \mathbf{v} \in H_0^1(V, \mathbb{R}^m)$. In particular, we can test the solution $\mathbf{v}$ against this particular function and obtain

$$\int_{B_r(x_0)} \phi^2 \nabla \mathbf{v} : \nabla \mathbf{v} + 2\phi \nabla \mathbf{v} : (\mathbf{v} \cdot \nabla \phi) \, \mathrm{d}x = 0.$$

From this, deducing (A.6) is just a matter of using Cauchy-Schwarz inequality and the properties of the cut-off function ϕ. ■

In the literature an estimate of type (A.6) is referred to as Caccioppoli inequality. Together with Sobolev imbedding and Poincaré inequality, one can then deduce the following estimate for harmonic functions due to Campanato [2]

$$\int_{B_\rho(x_0)} \left| \mathbf{v} - [\mathbf{v}]_{B_\rho(x_0)} \right|^2 \mathrm{d}x \le c \left(\frac{\rho}{r} \right)^{n+2} \int_{B_r(x_0)} \left| \mathbf{v} - [\mathbf{v}]_{B_r(x_0)} \right|^2 \mathrm{d}x. \tag{A.7}$$

Here $c = c(d, m) > 0$ denotes again an absolute constant and the assumptions on the radii of the concentric balls are the same as above. We refer to an estimate of type (A.7) as Campanato inequality.

As it turns out, for our purposes we will need appropriate L^p generalizations of these two inequalities. For instance, with respect to Campanato inequality we would like to have a suitable version even for the degenerate case of $1 < p < 2$. Fortunately, this particular question was already successfully dealt with in [27, 28]. More precisely, we are equipped with the following result.

Proposition A.3. *Let $V \subset \mathbb{R}^d$ be a domain and $\mathbf{v} \in W^{1,p}(V, \mathbb{R}^m)$ a weak solution to $\Delta \mathbf{v} = 0$ in V, where $1 < p < 2$. Then, for all $x_0 \in V$ and all $\rho < r < \mathrm{dist}(x_0, \partial V)$, the following generalized L^p-Campanato inequality holds true*

$$\int_{B_\rho(x_0)} \left| \mathbf{v} - [\mathbf{v}]_{B_\rho(x_0)} \right|^p \mathrm{d}x \le c \left(\frac{\rho}{r} \right)^{n+p} \int_{B_r(x_0)} \left| \mathbf{v} - [\mathbf{v}]_{B_r(x_0)} \right|^p \mathrm{d}x, \tag{A.8}$$

with the constant $c > 0$ depending only on the dimensions and the choice of p.

With respect to the classical Caccioppoli inequality (A.6) for harmonic functions, we will need a suitable generalization for the case $2 < p < \infty$. For this, recall that by means of the mean value equality, roughly speaking the L^∞-norm of a harmonic function is controlled by the corresponding L^1-norm. Having this in mind, it is now natural to consider an interpolation argument for the proof of

Proposition A.4. *Let $V \subset \mathbb{R}^d$ be a domain, $\mathbf{v} \in H^1(V,\mathbb{R}^m)$ a weak solution for Laplace's equation $\Delta \mathbf{v} = 0$ in V and $2 < p < \infty$. Then, for all $x_0 \in V$ and all $\rho < r < \operatorname{dist}(x_0, \partial V)$, the following generalized L^p-Caccioppoli inequality holds true*

$$\int_{B_\rho(x_0)} |\nabla \mathbf{v}|^p \,\mathrm{d}x \le c \left(\frac{r^{\alpha(d,p)}}{(r-\rho)^{\alpha(d,p)+1}} \right)^p \int_{B_r(x_0)} |\mathbf{v}|^p \,\mathrm{d}x, \tag{A.9}$$

with $\alpha(d,p) = d(p-2)/2$ and the constant $c > 0$ solely depending on p.

Proof. First of all, it is well known that actually $\mathbf{v} \in C^\infty(V,\mathbb{R}^m)$. Now, consider arbitrary real numbers $\rho \le s < t \le r$. By the mean value equality we know that for all $y \in B_s(x_0)$

$$|\nabla \mathbf{v}(y)| \le \frac{1}{|B_{t-s}(y)|} \int_{B_{t-s}(y)} |\nabla \mathbf{v}(x)| \,\mathrm{d}x \le \frac{c}{(t-s)^d} \|\nabla \mathbf{v}\|_{L^1(B_t(x_0))}.$$

On the other side, interpolation yields

$$\|\nabla \mathbf{v}\|_{L^p(B_s(x_0))} \le \|\nabla \mathbf{v}\|_{L^2(B_s(x_0))}^{2/p} \|\nabla \mathbf{v}\|_{L^\infty(B_s(x_0))}^{(p-2)/p}.$$

Together with the classical Caccioppoli inequality, these two bounds imply that

$$\begin{aligned}
\int_{B_s(x_0)} |\nabla \mathbf{v}|^p \,\mathrm{d}x &\le c \frac{1}{(t-s)^2} \int_{B_t(x_0)} |\mathbf{v}|^2 \,\mathrm{d}x \cdot \frac{1}{(t-s)^{d(p-2)}} \|\nabla \mathbf{v}\|_{L^1(B_t(x_0))}^{p-2} \\
&\le c \frac{r^{d(p-2)/p}}{(t-s)^2} \left(\int_{B_r(x_0)} |\mathbf{v}|^p \,\mathrm{d}x \right)^{\frac{2}{p}} \cdot \frac{r^{d(p-2)(p-1)/p}}{(t-s)^{d(p-2)}} \|\nabla \mathbf{v}\|_{L^p(B_t(x_0))}^{p-2} \\
&= c \frac{r^{d(p-2)}}{(t-s)^{d(p-2)+2}} \left(\|\mathbf{v}\|_{L^p(B_r(x_0))}^p \right)^{\frac{2}{p}} \left(\|\nabla \mathbf{v}\|_{L^p(B_t(x_0))}^p \right)^{\frac{p-2}{p}}.
\end{aligned}$$

whereat we also used Hölder inequality. An application of Young inequality eventually shows that

$$\int_{B_s(x_0)} |\nabla \mathbf{v}|^p \,\mathrm{d}x \le c \left(\frac{r^{\alpha(d,p)}}{(t-s)^{\alpha(d,p)+1}} \right)^p \int_{B_r(x_0)} |\mathbf{v}|^p \,\mathrm{d}x + \frac{p-2}{p} \int_{B_t(x_0)} |\nabla \mathbf{v}|^p \,\mathrm{d}x.$$

Since $0 < (p-2)/p < 1$, it remains to exploit the iteration scheme from Proposition A.1 to complete the proof. ■

An interpolation inequality Let $r > 0$, $(x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}$ and consider the parabolic space-time cylinder $Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0)$. It is not hard to establish the following well-known interpolation inequality which is used frequently in the text.

Proposition A.5. *Let* $\mathbf{u}\colon Q_r \to \mathbb{R}^3$ *be a vector-valued function with finite energy, i.e.* $\mathbf{u} \in L^{2,\infty}(Q_r) \cap L^2\big(t_0 - r^2, t_0; W^{1,2}(B_r(x_0), \mathbb{R}^3)\big)$. *(Here, we abbreviated* $Q_r = Q_r(x_0, t_0)$.*) Then, the following inequality is valid:*

$$\|\mathbf{u}\|^3_{L^3(Q_r)} \lesssim r^{\frac{1}{2}} \|\mathbf{u}\|^{\frac{3}{2}}_{L^{2,\infty}(Q_r)} \left(\|\nabla \mathbf{u}\|^2_{L^2(Q_r)} + \|\mathbf{u}\|^2_{L^{2,\infty}(Q_r)}\right)^{\frac{3}{4}}. \tag{A.10}$$

The proportionality factor implicit in this bound is given by an absolute constant.

Proof. Fix $t \in (t_0 - r^2, t_0)$ such that $\mathbf{u}(t) \in W^{1,2}(B_r(x_0), \mathbb{R}^3)$. By Hölder's inequality, we obtain the bound

$$\begin{aligned}\int_{B_r(x_0)} |\mathbf{u}(t)|^3 \,\mathrm{d}x &= \int_{B_r(x_0)} |\mathbf{u}(t)|^{\frac{3}{2}} |\mathbf{u}(t)|^{\frac{3}{2}} \,\mathrm{d}x \\ &\le \left(\int_{B_r(x_0)} |\mathbf{u}(t)|^2 \,\mathrm{d}x\right)^{\frac{3}{4}} \left(\int_{B_r(x_0)} |\mathbf{u}(t)|^6 \,\mathrm{d}x\right)^{\frac{1}{4}}.\end{aligned}$$

Furthermore, a version of the Gagliardo–Nirenberg inequality ensures that

$$\|\mathbf{u}(t)\|_{L^6(B_r(x_0))} \lesssim \left(\int_{B_r(x_0)} |\nabla \mathbf{u}(t)|^2 + r^{-2} |\mathbf{u}(t)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}.$$

Hence,

$$\int_{B_r(x_0)} |\mathbf{u}(t)|^3 \,\mathrm{d}x \lesssim \left(\int_{B_r(x_0)} |\mathbf{u}(t)|^2 \,\mathrm{d}x\right)^{\frac{3}{4}} \left(\int_{B_r(x_0)} |\nabla \mathbf{u}(t)|^2 + r^{-2} |\mathbf{u}(t)|^2 \,\mathrm{d}x\right)^{\frac{3}{4}}.$$

As this last bound holds for a.e. $t \in (t_0 - r^2, t_0)$, we can integrate in time and obtain (after another application of Hölder's inequality) the bound

$$\int_{Q_r} |\mathbf{u}|^3 \,\mathrm{d}x \lesssim r^{\frac{1}{2}} \|\mathbf{u}\|^{\frac{3}{2}}_{L^{2,\infty}(Q_r)} \left(\int_{Q_r} |\nabla \mathbf{u}|^2 + r^{-2} |\mathbf{u}|^2 \,\mathrm{d}x\right)^{\frac{3}{4}}.$$

This concludes the proof. ■

A result from the L^p-theory for the bi-harmonic equation As it turns out, the following result for the bi-harmonic equation will be of great importance. We refer to [23] for a proof.

Proposition A.6. *Let $U \subset \mathbb{R}^d$ be an open and bounded set with boundary ∂U in C^2 and let $1 < p < \infty$. For every $F \in W_0^{2,p}(U)'$ there exists a uniquely determined function $u \in W_0^{2,p}(U)$ such that for every $\phi \in C_c^\infty(U)$*

$$\int_U \Delta u \Delta \phi \,\mathrm{d}x = F(\phi).$$

Furthermore, in this situation the following bound does hold

$$\|u\|_{W_0^{2,p}(U)} \lesssim \|F\|_{W_0^{2,p}(U)'}. \tag{A.11}$$

Hausdorff measures Consider $\mathbb{R}^d$ and let $\alpha > 0$, $0 < \delta \le 1$. Furthermore, we denote by $\mathfrak{K}$ the set of all compact subsets of $\mathbb{R}^d$. Then, for all $A \subset \mathbb{R}^d$, we define outer measures $\mathscr{H}_{\alpha,\delta}$ and $\mathscr{H}_\alpha$ by means of

$$\mathscr{H}_{\alpha,\delta}(A) := \inf\left\{ \sum_{n=1}^{\infty} \operatorname{diam}(A_n)^\alpha : \ A_n \in \mathfrak{K}, \operatorname{diam}(A_n) \le \delta, \ A \subset \bigcup_{n=1}^{\infty} A_n \right\},$$
$$\mathscr{H}_\alpha(A) := \sup_{0<\delta\le 1} \mathscr{H}_{\alpha,\delta}(A).$$

We call $\mathscr{H}_\alpha$ the α-dimensional outer Hausdorff measure. Let us denote by $\mathfrak{A}_\alpha$ the set of all $\mathscr{H}_\alpha$-measurable subsets of $\mathbb{R}^d$. The general construction theme for measures due to Carathéodory then guarantees that the triple $(\mathbb{R}^d, \mathfrak{A}_\alpha, \mathscr{H}_\alpha|\mathfrak{A}_\alpha)$ is indeed a measure space. Furthermore, it is a well-known fact that all Borel-measurable subsets of $\mathbb{R}^d$ are contained in the σ-algebra $\mathfrak{A}_\alpha$. For proofs of all these statements, we refer to [3].

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